Original Scientific Paper | Received: 24-11-2017 | Accepted: 06-01-2018

Standard Parallel and Secant Parallel in Azimuthal Projections

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*Since author is the journal's managing editor, the peer review process and independent editorial decision were performed by an external editor, Professor Emeritus Nedjeljko Frančula.
We thank Professor Emeritus Nedjeljko Frančula for his help in addressing potential managing editor's conflict of interest.

Abstract. It is commonly assumed that standard parallels and parallels that appear as the intersections of a developable surface and a sphere or ellipsoid coincide. This paper shows that this is not true of azimuthal projections which are equidistant along meridians, equidistant along parallels (orthographic), and equal-area, because there is no standard parallel at all in such projections. Only some azimuthal conformal (stereographic) projections have standard parallel, in which case it coincides with secant parallel.

Keywords: map projection, azimuthal projection, standard parallel, secant parallel

1 Introduction

Map projections are commonly approached as mapping onto developed surfaces; cylindrical projections onto the lateral surface of a cylinder, conic projections onto the lateral surface of a cone, and azimuthal projections onto a plane. Kessler (2017) researched map projections and found that 21 of 23 general cartographic textbooks published between 1902 and 2014 contained images of cones, cylinders or planes, and the term "developmental surface".

Sometimes, azimuthal projections are called planar projections, because it is easier to relate to the developable surface concept (Slocum et al. 2009). This is not advisable, because all map projections are mappings onto a plane.

Standard parallels are mapped with no deformations. More details about these, with the pertinent mathematical expressions, are given in Chapter 2 of this paper.

If the intermediate developable surface intersects the Earth's sphere or ellipsoid, it is related to the *secant projections*. The intersection of the developable surface and the Earth's sphere or ellipsoid, e.g. *secant parallel*, is considered a standard parallel without proof (see e.g. Behrmann 1910, Richardus and Adler 1972, Snyder and Voxland 1989, USGS 2000, NPTEL 2007, Slocum et al. 2009, Wayback Machine 2014, Wikimedia Commons 2016, Albrecht 2017, ESRI 2017, Geokov 2017, Van Sickle 2017). This is usually, but not always taken to be the case. In this paper, we consider azimuth projections of a sphere, and for an ellipsoid it can be done analogously. The secant parallel in azimuthal projections is the parallel of latitude at which the plane of the projection or map placed parallel to the equatorial plane cuts the sphere.

The following approach is described in *Thematic Cartography and Geovisualization* (Slocum et al. 2009). "Secant lines and points of tangency each have the same scale as the principal scale of the reference globe. Thus, secant lines are called standard lines, and points of tangency are called standard points. All other lines and points will have either a larger or a smaller scale than the principal map scale of the reference globe. Figure 8.12 illustrates the concept of a standard line and its impact on scale variation across a map. In the figure, a portion of the reference globe is represented by the dashed line, and the developable surface is represented by the solid line of gray values. Note that the developable surface cuts the reference globe, creating two standard lines ...". It is difficult to imagine that the plane would cut the sphere into two (standard) circles! In addition, the secant lines are generally distorted (the local linear scale factor is different from 1) and therefore we cannot call them standard lines, or lines without distortion.

Izvorni znanstveni rad | Primljeno: 24-11-2017 | Prihvaćeno: 06-01-2018

Standardna paralela i presječna paralela kod azimutnih projekcija

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*S obzirom na to da je autor izvršni urednik ovoga časopisa, recenziranje je obavio i neovisnu uredničku odluku donio vanjski urednik prof. emer. Nedjeljko Frančula.
Zahvaljujemo prof. emer. Nedjeljku Frančuli na pomoći vezanoj uz potencijalni sukob interesa izvršnog urednika.

Sažetak. Uvriježeno je mišljenje da se standardne paralele i presječne paralele poklapaju. U ovome radu pokazuje se da to nije istina ni za jednu uspravnu azimutnu ekvidistantnu uzduž meridijana, uspravnu azimutnu ekvidistantnu uzduž paralela (ortografsku) ni uspravnu azimutnu ekvivalentnu projekciju jer te projekcije uopće nemaju standardnih paralela. Samo neke uspravne azimutne konformne (stereografske) projekcije imaju standardnu paralelu koja je istodobno i presječna paralela.

Ključne riječi: kartografske projekcije, standardne paralele, presječne paralele

1. Uvod

U kartografskoj literaturi je uobičajeno da se kartografske projekcije tumače kao projekcije na razvojne plohe – cilindrične projekcije kao projekcije na plašt cilindra, konusne projekcije kao projekcije na plašt konusa, a azimutne projekcije kao projekcije u ravninu. Kessler (2017) je istražujući pristup kartografskim projekcijama ustanovio da 21 od 23 kartografska udžbenika sadrži slike konusa, cilindara ili ravnina i izraz "razvojna ploha". Riječ je o udžbenicima iz razdoblja 1902–2014.

Ponekad se azimutne projekcije nazivaju i ravninskim projekcijama kako bi se bolje uklopile u klasifikaciju projekcija na temelju posrednih razvojnih ploha (Slocum et al. 2009). To nije dobro jer su sve kartografske projekcije preslikavanja u ravninu, a i klasifikacija projekcija na temelju razvojnih ploha je vjerojatno nepotrebna.

Standardne paralele su one paralele koje se u projekciji preslikavaju bez ikakvih deformacija. Detaljnije o njima i s odgovarajućim matematičkim izrazima čitatelj će naći u 2. poglavlju ovoga rada.

Ako posredne razvojne plohe sijeku Zemljinu sferu ili elipsoid, govori se o *sjekućim projekcijama*. Presjeci tih ploha i Zemljine sfere ili elipsoida, npr. *presječne paralele*, poistovjećuju se u literaturi sa standardnim paralelama bez dokaza (vidjeti npr. Behrmann 1910, Richardus i Adler 1972, Snyder i Voxland 1989, USGS 2000, NPTEL 2007, Slocum et al. 2009, Wayback Machine 2014, Wikimedia Commons 2016, Albrecht 2017, ESRI 2017, Geokov 2017, Van Sickle 2017). To se redovito uzima kao očigledna činjenica premda nije uvijek tako. U ovome radu bavimo se azimutnim projekcijama sfere, a za elipsoid se može postupiti na analogan način. Presječna paralela kod azimutnih projekcija je ona paralela u kojoj ravnina projekcije, odnosno karta položena paralelno s ekvatorskom ravninom siječe sferu.

Na primjer u knjizi *Thematic Cartography and Geovisualization* (Slocum i dr. 2009) imamo ovakav pristup: "*Secant lines and points of tangency each have the same scale as the principal scale of the reference globe. Thus, secant lines are called standard lines, and points of tangency are called standard points. All other lines and points will have either a larger or a smaller scale than the principal map scale of the reference globe. Figure 8.12 illustrates the concept of a standard line and its impact on scale variation across a map. In the figure, a portion of the reference globe is represented by the dashed line, and the developable surface is represented by the solid line of gray values. Note that the developable surface cuts the reference globe, creating two standard lines ...*" Teško je zamisliti da ravnina siječe sferu u dvije (standardne) kružnice! Osim toga, sjekuće linije općenito nisu bez deformacija pa ih stoga ne možemo zvati linijama bez deformacija, tj. standardnim linijama.

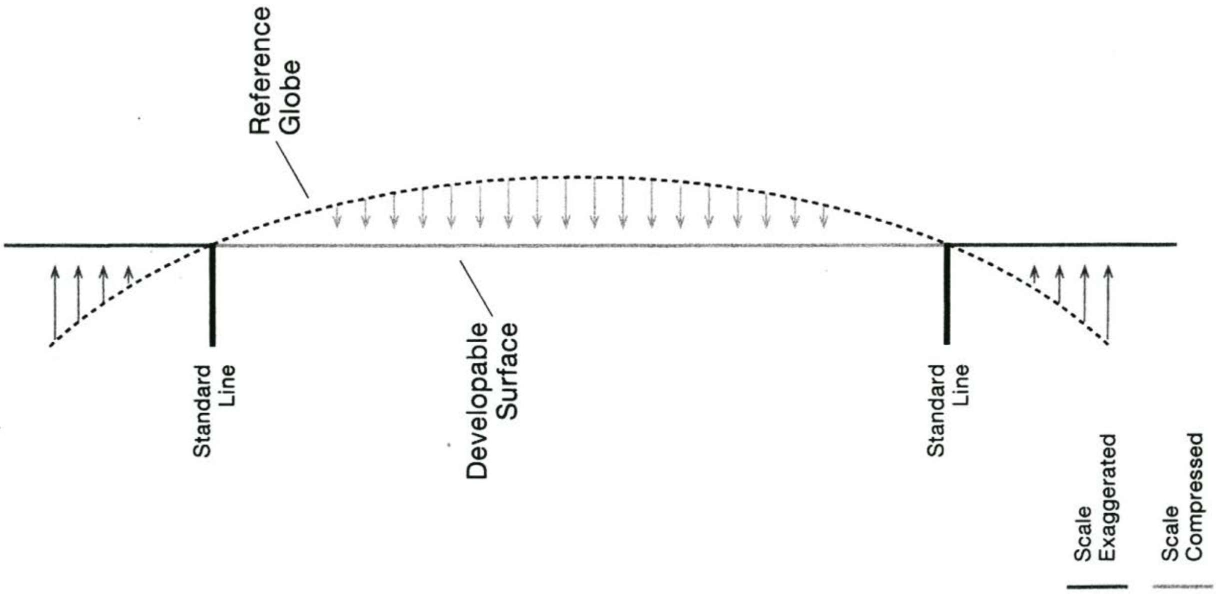

Fig. 1 Standard parallels are considered the same as secant parallels, in which the developing surface cuts the sphere. Source: *Thematic Cartography and Geovisualization* (Slocum et al. 2001, p. 139, Fig. 8.12). In this paper, we show that this is generally not correct, i.e. that there are common azimuthal map projections without standard parallels, and consequently without secant parallels

Slika 1. Na slici su standardne paralele poistovjećene s paralelama u kojima razvojna ploha (u našem slučaju ravnina) siječe sferu. Izvor: *Thematic Cartography and Geovisualization* (Slocum i dr. 2001, str. 139, slika 8.12). U ovome radu dokazuje se da to općenito ne vrijedi.

The same book (Slocum et al. 2009, page 138) also says: "In the secant case of the cylindrical and conic map projections (Figure 8.9.B and D), there are two secant lines, whereas in the case of the planar projection, there is one secant line (Figure 8.9F)," and in the footnote on the same page, "In the strictest sense, a secant case for the planar class is not possible, but it is included here for conceptual completeness." Does the theory of map projection need something that does not exist? Of course we do not need it. On the other hand, this statement is incorrect because there are azimuthal projections with secant parallels, as this paper will prove

First, let us define the geographic parameterization of a sphere with a radius $R>0$ and centre in the origin of the coordinate system as mapping

$$X = R\cos\varphi\cos\lambda, Y = R\cos\varphi\sin\lambda, Z = R\sin\varphi \quad (1)$$

where

$\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\lambda \in [-\pi, \pi]$. Parameter φ is the geographic latitude, while λ is the geographic longitude, as usual. It is not difficult to obtain the first differential form of mapping (1) as

$$ds^2 = R^2 d\varphi^2 + R^2 \cos^2\varphi d\lambda^2. \quad (2)$$

A map projection is usually defined as mapping into a plane by using the formulae

$$x = x(\varphi, \lambda) \ , \ y = y(\varphi, \lambda), \quad (3)$$

where x and y are coordinates in a rectangular (mathematical, right oriented) coordinate system in the plane, while $\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\lambda \in [-\pi, \pi]$. The first differential form of the mapping (3) is

$$ds'^2 = Ed\varphi^2 + 2Fd\varphi d\lambda + Gd\lambda^2 , \quad (4)$$

where the coefficients are

$$E = \left(\frac{\partial x}{\partial\varphi}\right)^2 + \left(\frac{\partial y}{\partial\varphi}\right)^2, \ F = \frac{\partial x}{\partial\varphi}\frac{\partial x}{\partial\lambda} + \frac{\partial y}{\partial\varphi}\frac{\partial y}{\partial\lambda}, \quad (5)$$

$$G = \left(\frac{\partial x}{\partial\lambda}\right)^2 + \left(\frac{\partial y}{\partial\lambda}\right)^2 .$$

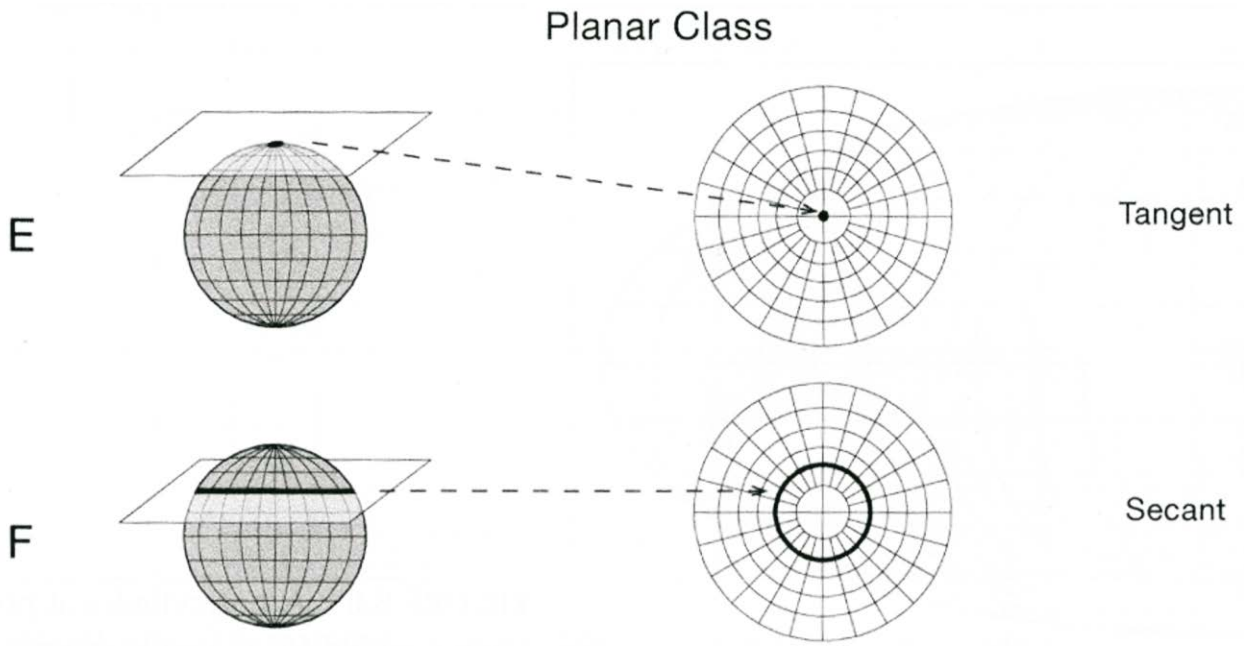

Fig. 2 Illustration of azimuthal projections. Source: *Thematic Cartography and Geovisualization* (Slocum et al. 2001, p. 136, part of Fig. 8.9). (In the book it is noted that the case of secant planar projections (depicted by letter F) is impossible, but is included for conceptual completeness!

Slika 2. Ilustracija za azimutne projekcije. Izvor: *Thematic Cartography and Geovisualization* (Slocum i dr. 2001, str. 136, dio slike 8.9). U tekstu se napominje da slučaj sjekuće projekcije (na slici označeno slovom F) nije moguć, ali je uključen radi koncepcijske potpunosti!

U istoj knjizi (Slocum i dr. 2009, page 138) možemo pročitati i ovo: "*In the secant case of the cylindrical and conic map projections* (*Figure 8.9.B and D*), *there are two secant lines, whereas in the case of the planar projection, there is one secant line* (*Figure 8.9F*)", and in the footnote at the same page: "*In the strictest sense, a secant case for the planar class is not possible, but it is included here for conceptual completeness.*" Treba li teoriji kartografskih projekcija nešto što ne postoji? Naravno da ne treba. S druge strane, taj navod nije točan jer postoje azimutne projekcije s presječnim paralelama. O tome se čitatelj može uvjeriti u nastavku ovog članka.

Najprije definirajmo geografsku parametrizaciju sfere polumjera *R>0* sa središtem u ishodištu koordinatnog sustava kao preslikavanje definirano formulama

$$X = R\cos\varphi\cos\lambda,\ Y = R\cos\varphi\sin\lambda,\ Z = R\sin\varphi \qquad (1)$$

$$\varphi \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right],\ \lambda \in [-\pi,\pi]$$

Pri tom zovemo φ geografskom širinom, a λ geografskom dužinom. Nije teško izvesti da je prva diferencijalna forma tog preslikavanja

$$ds^2 = R^2 d\varphi^2 + R^2\cos^2\varphi\, d\lambda^2 \qquad (2)$$

Kartografsku projekciju definiramo kao preslikavanje u ravninu zadano formulama

$$x = x(\varphi,\lambda)\ , y = y(\varphi,\lambda)\,. \qquad (3)$$

Pri tome su x i y koordinate točke u pravokutnom (matematičkom, desnom) koordinatnom sustavu u ravnini, a $\varphi \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, $\lambda \in [-\pi,\pi]$. Za takvo preslikavanje je prva diferencijalna forma

$$ds'^2 = Ed\varphi^2 + 2Fd\varphi d\lambda + Gd\lambda^2\,, \qquad (4)$$

gdje su koeficijenti

$$E = \left(\frac{\partial x}{\partial\varphi}\right)^2 + \left(\frac{\partial y}{\partial\varphi}\right)^2,\ F = \frac{\partial x}{\partial\varphi}\frac{\partial x}{\partial\lambda} + \frac{\partial y}{\partial\varphi}\frac{\partial y}{\partial\lambda},$$

$$G = \left(\frac{\partial x}{\partial\lambda}\right)^2 + \left(\frac{\partial y}{\partial\lambda}\right)^2. \qquad (5)$$

If a map projection is defined in a polar coordinate system, as is usual in azimuthal map projections, then the relation between the rectangular coordinates *x, y* and the polar coordinates ρ, θ is as follows:

$$x = \rho \sin\theta \ , \quad y = \rho\cos\theta \tag{6}$$

with

$$\rho = \rho(\varphi) \ , \quad \theta = \lambda - \lambda_0 \ , \tag{7}$$

where $\lambda_0 \in [-\pi, \pi]$ is a constant of projection. It is easy to compute partial derivatives

$$\begin{aligned} \frac{\partial x}{\partial\varphi} &= \frac{\partial\rho}{\partial\varphi}\sin\theta \quad & \frac{\partial y}{\partial\varphi} &= \frac{\partial\rho}{\partial\varphi}\cos\theta \\ \frac{\partial x}{\partial\lambda} &= \rho\cos\theta \quad & \frac{\partial y}{\partial\lambda} &= -\rho\sin\theta \end{aligned} \tag{8}$$

and after (5), we get

$$E = \left(\frac{\partial\rho}{\partial\varphi}\right)^2, \ F = 0, \ G = \rho^2. \tag{9}$$

The local linear scale factor *c* for mapping a sphere into a plane is defined by using the following relation

$$c^2 = \frac{ds'^2}{ds^2} = \frac{Ed\varphi^2 + 2Fd\varphi d\lambda + Gd\lambda^2}{R^2 d\varphi^2 + R^2\cos^2\varphi d\lambda^2} \ , \tag{10}$$

which can also be written as

$$c^2(\alpha) = \frac{E}{R^2}\cos^2\alpha + \frac{F}{R^2\cos\varphi}\sin 2\alpha + \frac{G}{R^2\cos^2\varphi}\sin^2\alpha \tag{11}$$

where

$$\tan\alpha = \frac{\cos\varphi d\lambda}{d\varphi} \ . \tag{12}$$

The poles are singular points of geographic parameterization (1) and therefore expressions (10) and (11) and all subsequent ones should be interpreted in the poles as limiting cases when $\varphi \to \frac{\pi}{2}$ or $\varphi \to -\frac{\pi}{2}$.

If $\alpha = 0$ or, more generally, $\alpha = z\pi, z \in Z$, where *Z* denotes the set of all integers, then the local linear scale factor along a meridian is

$$c = h = \frac{\sqrt{E}}{R} \ , \tag{13}$$

and if $\alpha = \frac{\pi}{2}$ or, more generally, $\alpha = \frac{\pi}{2} + z\pi, z \in Z$, then the local linear scale factor along a parallel is given by

$$c = k = \frac{\sqrt{G}}{R\cos\varphi} \ . \tag{14}$$

Next, we will find the extremes of the local linear scale factor. For this purpose, let us denote

$$\lambda(\alpha) = c^2(\alpha) \ . \tag{15}$$

Using the substitutions

$$\sin^2\alpha = \frac{1-\cos 2\alpha}{2}, \quad \cos^2\alpha = \frac{1+\cos 2\alpha}{2} \tag{16}$$

$$K = \frac{1}{R^2\cos^2\varphi}\sqrt{\left(E\cos^2\varphi - G\right)^2 + 4F^2\cos^2\varphi}, \tag{17}$$

$$\sin 2\vartheta = \frac{2F}{KR^2\cos\varphi} \ , \quad \cos 2\vartheta = \frac{E\cos^2\varphi - G}{KR^2\cos^2\varphi} \tag{18}$$

and

$$t = \alpha - \vartheta \tag{19}$$

we get

$$\lambda(\alpha) = \frac{E\cos^2\varphi + G}{2R^2\cos^2\varphi} + \frac{K}{2}\cos 2t \tag{20}$$

from which the extremes can be read:

$$\lambda_{max} = \frac{E\cos^2\varphi + G}{2R^2\cos^2\varphi} + \frac{K}{2} \tag{21}$$

for $t = z\pi, z \in Z$, or

$$\alpha = \vartheta + z\pi, z \in Z \tag{22}$$

and

$$\lambda_{min} = \frac{E\cos^2\varphi + G}{2R^2\cos^2\varphi} - \frac{K}{2} \tag{23}$$

for $t = \frac{\pi}{2} + z\pi, z \in Z$, or

$$\alpha = \vartheta + \frac{\pi}{2} + z\pi, z \in Z \ . \tag{24}$$

In the previous formulae, the function $\lambda = \lambda(\alpha)$ does not refer to latitude, which is also indicated by the Greek letter λ, since from the context we can clearly see what it means. These features of the extreme values of the function $\lambda = \lambda(\alpha)$ are interesting:

$$\lambda_{min} + \lambda_{max} = \frac{E\cos^2\varphi + G}{R^2\cos^2\varphi} \tag{25}$$

$$\lambda_{min}\lambda_{max} = \frac{EG - F^2}{R^4\cos^2\varphi} \tag{26}$$

which means that λ_{min} and λ_{max} are the solutions of the quadratic equation

$$\lambda^2 - \frac{E\cos^2\varphi + G}{R^2\cos^2\varphi}\lambda + \frac{EG - F^2}{R^4\cos^2\varphi} = 0 \ . \tag{27}$$

Ako je projekcija zadana u polarnom koordinatnom sustavu, kao što je to uobičajeno kod azimutnih projekcija, tada je veza između pravokutnih x, y i polarnih koordinata ρ, δ ova:

$$x=\rho\sin\theta \quad y=\rho\cos\theta \tag{6}$$

pri čemu je

$$\rho=\rho(\varphi)\ ,\ \theta=\lambda-\lambda_0\ , \tag{7}$$

a $\lambda_0\in[-\pi,\pi]$ konstanta projekcije. Lako izračunamo

$$\frac{\partial x}{\partial\varphi}=\frac{\partial\rho}{\partial\varphi}\sin\theta,\qquad \frac{\partial y}{\partial\varphi}=\frac{\partial\rho}{\partial\varphi}\cos\theta\,,$$
$$\frac{\partial x}{\partial\lambda}=\rho\cos\theta,\qquad \frac{\partial y}{\partial\lambda}=-\rho\sin\theta \tag{8}$$

i zatim prema (5) dobijemo

$$E=\left(\frac{\partial\rho}{\partial\varphi}\right)^2\ ,\ F=0,\quad G=\rho^2. \tag{9}$$

Faktor lokalnog linearnog mjerilo c za preslikavanje sfere definira se ovako

$$c^2=\frac{ds'^2}{ds^2}=\frac{Ed\varphi^2+2Fd\varphi d\lambda+Gd\lambda^2}{R^2d\varphi^2+R^2\cos^2\varphi d\lambda^2}, \tag{10}$$

što se može napisati i u obliku

$$c^2(\alpha)=\frac{E}{R^2}\cos^2\alpha+\frac{F}{R^2\cos\varphi}\sin2\alpha+\frac{G}{R^2\cos^2\varphi}\sin^2\alpha \tag{11}$$

gdje je

$$\tan\alpha=\frac{\cos\varphi d\lambda}{d\varphi}\ . \tag{12}$$

Polovi su singularne točke geografske parametrizacije (1) i zbog toga izraze (10) i (11) i sve koji iz njih slijede treba u polovima interpretirati kao granične vrijednosti kad $\varphi\to\frac{\pi}{2}$, odnosno kad $\varphi\to-\frac{\pi}{2}$.

Ako je $\alpha=0$ ili općenitije $\alpha=z\pi, z\in Z,$ gdje je sa Z označen skup svih cijelih brojeva, onda je faktor lokalnog linearnog mjerila uzduž meridijana

$$c=h=\frac{\sqrt{E}}{R}\ , \tag{13}$$

a ako je $\alpha=\frac{\pi}{2}$ ili općenitije $\alpha=\frac{\pi}{2}+z\pi, z\in Z$, onda je faktor lokalnog linearnog mjerila uzduž paralela

$$c=k=\frac{\sqrt{G}}{R\cos\varphi}. \tag{14}$$

Sad ćemo istražiti ekstremne vrijednosti faktora lokalnog linearnog mjerila. U tu svrhu označimo

$$\lambda(\alpha)=c^2(\alpha) \tag{15}$$

i uz supstitucije

$$\sin^2\alpha=\frac{1-\cos2\alpha}{2}\ ,\ \cos^2\alpha=\frac{1+\cos2\alpha}{2} \tag{16}$$

$$K=\frac{1}{R^2\cos^2\varphi}\sqrt{\left(E\cos^2\varphi-G\right)^2+4F^2\cos^2\varphi}\,, \tag{17}$$

$$\sin2\vartheta=\frac{2F}{KR^2\cos\varphi}\ ,\ \cos2\vartheta=\frac{E\cos^2\varphi-G}{KR^2\cos^2\varphi} \tag{18}$$

i

$$t=\alpha-\vartheta \tag{19}$$

možemo dobiti

$$\lambda(\alpha)=\frac{E\cos^2\varphi+G}{2R^2\cos^2\varphi}+\frac{K}{2}\cos2t \tag{20}$$

odakle se mogu pročitati ekstremne vrijednosti:

$$\lambda_{\max}=\frac{E\cos^2\varphi+G}{2R^2\cos^2\varphi}+\frac{K}{2} \tag{21}$$

što se postiže za $t=z\pi, z\in Z,$
odnosno za

$$\alpha=\vartheta+z\pi, z\in Z \tag{22}$$

i

$$\lambda_{\min}=\frac{E\cos^2\varphi+G}{2R^2\cos^2\varphi}-\frac{K}{2} \tag{23}$$

što se postiže za $t=\frac{\pi}{2}+z\pi, z\in Z,$
odnosno za

$$\alpha=\vartheta+\frac{\pi}{2}+z\pi, z\in Z\ . \tag{24}$$

U prethodnim formulama oznaku funkcije $\lambda=\lambda(\alpha)$ ne treba dovoditi u vezu s geografskom dužinom koja se također označava grčkim slovom λ, ali se iz konteksta može jasno razabrati o čemu je riječ. Zanimljiva su ova svojstva ekstremnih vrijednosti funkcije $\lambda=\lambda(\alpha)$

$$\lambda_{\min}+\lambda_{\max}=\frac{E\cos^2\varphi+G}{R^2\cos^2\varphi} \tag{25}$$

$$\lambda_{\min}\lambda_{\max}=\frac{EG-F^2}{R^4\cos^2\varphi} \tag{26}$$

što znači da su $\lambda_{\min}$ i $\lambda_{\min}$ rješenja kvadratne jednadžbe

$$\lambda^2-\frac{E\cos^2\varphi+G}{R^2\cos^2\varphi}\lambda+\frac{EG-F^2}{R^4\cos^2\varphi}=0\ . \tag{27}$$

This quadratic equation can be written as

$$\begin{vmatrix} \frac{E}{R^2}-\lambda & \frac{F}{R^2\cos\varphi} \\ \frac{F}{R^2\cos\varphi} & \frac{G}{R^2\cos^2\varphi}-\lambda \end{vmatrix}=0 \qquad (28)$$

and interpreted as searching the eigenvalues of the quadratic form (11). Formula (20) can be transformed into

$$\lambda(\alpha)=\lambda_{max}\cos^2 t+\lambda_{min}\sin^2 t= =\lambda_{max}\cos^2(\alpha-\vartheta)+\lambda_{min}\sin^2(\alpha-\vartheta) \ . \qquad (29)$$

It is not difficult to see that $\sqrt{\lambda_{max}}$ and $\sqrt{\lambda_{min}}$ are the semi-axes of Tissot's indicatrix, or the ellipse of distortion of the map projection.

In a special case, when

$$F=0 \qquad (30)$$

i.e. when the images of meridians and parallels intersect at right angles in the plan of projection, applying assumption (30), we get

$$c^2(\alpha)=\lambda(\alpha)=\frac{E}{R^2}\cos^2\alpha+\frac{G}{R^2\cos^2\varphi}\sin^2\alpha \ . \qquad (31)$$

In this case, if $E\cos^2\varphi=G$ then $\lambda=\frac{E}{R^2}=\frac{G}{R^2\cos^2\varphi}$ does not depend on angle .

If $E\cos^2\varphi>G$ then $\lambda_{max}=\frac{E}{R^2}$ for $\alpha=z\pi, z\in Z$ and $\lambda_{min}=\frac{G}{R^2\cos^2\varphi}$ for $\alpha=\frac{\pi}{2}+z\pi, z\in Z$.

If $E\cos^2\varphi<G$ then $\lambda_{min}=\frac{E}{R^2}$ for $\alpha=z\pi, z\in Z$ and $\lambda_{max}=\frac{G}{R^2\cos^2\varphi}$ for $\alpha=\frac{\pi}{2}+z\pi, z\in Z$.

2 Standard Parallels

We say that there are no distortions in a point due to the map projection, or that the distortion is zero, if

$$c(\alpha)=1 \ , \ \text{for each} \qquad (32)$$

where $c(\alpha)$ is defined by (11). This requirement is clearly equivalent to the condition

$$\lambda(\alpha)=1 \ , \ \text{for each } \alpha, \qquad (33)$$

or conditions

$$\lambda_{min}=\lambda_{max}=1 \ . \qquad (34)$$

Tissot's indicatrix was transformed into a unit circle. A line without distortions, or a line along which the distortion is zero, or a standard line, is a curve satisfying condition (32), (33) or (34) at each point. A *standard parallel* is a parallel without distortion, i.e. a parallel satisfying condition (32), (33) or (34) at each point.

3 Azimuthal Projections

First, we will look at several definitions of azimuthal projections. According to the National Atlas of the United States of America, which was withdrawn from the Internet at the end of 2014, but is still available (Wayback Machine 2014), an azimuthal projection is thus defined: "Azimuthal projection – a map projection in which the direction from a given central point to any other point is shown correctly. Also called a zenithal projection". Interestingly, the same atlas gives this definition: "Planar projection – a map projection resulting from conceptual projection of the Earth onto a tangent or secant plane. Usually, a planar projection is the same as an azimuthal projection." The latter definition of a plane projection makes no sense, because all map projections are planar, but any planar projection is not necessarily azimuthal projection.

Similarly, in *Thematic Cartography and Geovisualization* (Slocum et al. 2009) we find this approach: "The planar class of projections results from positioning the developable surface of a plane next to the reference globe and projecting the landmasses and graticule onto the plane". And in the footnote: "Some texts refer to the planar class of projections as azimuthal. We have chosen to use planar because it is easier to relate to the developable surface concept". In addition to the remarks in the previous paragraph, we should add that the use of the plane as a developable surface is confusing, because developing the plane itself is certainly not a particularly wise approach.

The USGS publication on map projections differentiates azimuthal projections by projection, into geometric and mathematical (USGS 2000). This is rather strange, because geometry is a branch of mathematics. But it seems to be a common approach in the US literature on map projections (see e.g. Snyder 1987, Snyder and Voxland 1989).

Normal aspect azimuthal projection is mapping given by

$$\rho=Rf(\varphi) \ , \ \theta=\lambda-\lambda_0 \ , \qquad (35)$$

where $\varphi\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, $\lambda\in[-\pi,\pi]$, the constants are $R>0$ and $\lambda_0\in[-\pi,\pi]$, and $\rho=Rf(\varphi)$ is continuous, with

Ta se kvadratna jednadžba može se napisati i u obliku

$$\begin{vmatrix} \frac{E}{R^2}-\lambda & \frac{F}{R^2\cos\varphi} \\ \frac{F}{R^2\cos\varphi} & \frac{G}{R^2\cos^2\varphi}-\lambda \end{vmatrix} = 0 \quad (28)$$

te protumačiti kao traženje svojstvenih vrijednosti kvadratne forme (11). Formula (20) može se napisati

$$\lambda(\alpha) = \lambda_{max}\cos^2 t + \lambda_{min}\sin^2 t = \\ = \lambda_{max}\cos^2(\alpha-\vartheta) + \lambda_{min}\sin^2(\alpha-\vartheta). \quad (29)$$

Nije teško prepoznati da su $\sqrt{\lambda_{max}}$ i $\sqrt{\lambda_{min}}$ poluosi Tissotove indikatrise, odnosno elipse deformacija promatrane kartografske projekcije. U posebnom slučaju kad je

$$F = 0 \quad (30)$$

tj. kad se slike meridijana i paralela u projekciji sijeku pod pravim kutom, formula (11) prelazi u

$$c^2(\alpha) = \lambda(\alpha) = \frac{E}{R^2}\cos^2\alpha + \frac{G}{R^2\cos^2\varphi}\sin^2\alpha . \quad (31)$$

U tom slučaju ako je $E\cos^2\varphi = G$ onda

$\lambda = \frac{E}{R^2} = \frac{G}{R^2\cos^2\varphi}$ ne ovisi o kutu α.

Ako je $E\cos^2\varphi > G$ onda je $\lambda_{max} = \frac{E}{R^2}$ za $\alpha = z\pi, z \in Z$

i $\lambda_{min} = \frac{G}{R^2\cos^2\varphi}$ za $\alpha = \frac{\pi}{2} + z\pi, z \in Z$.

Ako je $E\cos^2\varphi < G$ onda je $\lambda_{min} = \frac{E}{R^2}$ za

$\alpha = \frac{\pi}{2} + z\pi, z \in Z$ i $\lambda_{max} = \frac{G}{R^2\cos^2\varphi}$ za $\alpha = z\pi, z \in Z$.

2. Standardne paralele

Reći ćemo da u nekoj točki nema deformacija zbog kartografske projekcije, odnosno da je deformacija jednaka 0, ako je

$$c(\alpha) = 1, \text{ za svaki } \alpha \quad (32)$$

gdje je $c(\alpha)$ definirano u (11). Taj je zahtjev očito ekvivalentan uvjetu

$$\lambda(\alpha) = 1, \text{ za svaki } \alpha , \quad (33)$$

odnosno uvjetima

$$\lambda_{min} = \lambda_{max} = 1 \quad (34)$$

koji nam govore da je riječ o Tissotovoj indikatrisi koja je prešla u jediničnu kružnicu. Krivulja bez deformacija ili standardna linija, odnosno krivulja uzduž koje je deformacija 0 je krivulja za koju u svakoj točki vrijedi (32), (33) ili (34). *Standardna paralela* je paralela bez deformacija, tj. paralela za koju u svakoj točki vrijedi (32), (33) ili (34).

3. Azimutne projekcije

Najprije ćemo se osvrnuti na nekoliko definicija azimutnih projekcija. U The National Atlas of the United States of America, koji je povučen s interneta krajem 2014. godine, ali svejedno dostupan (Wayback Machine 2014), imamo ovu definiciju azimutnih projekcija: "*Azimuthal projection – A map projection in which the direction from a given central point to any other point is shown correctly. Also called a zenithal projection*". Zanimljivo je da u istom atlasu imamo i ovu definiciju: "*Planar projection – A map projection resulting from conceptual projection of the Earth onto a tangent or secant plane. Usually, a planar projection is the same as an azimuthal projection.*" Posljednja definicija ravninske projekcije nema smisla jer je svaka kartografska projekcija ravninska, a svaka takva projekcija nije azimutna.

Slično i u knjizi *Thematic Cartography and Geovisualization* (Slocum i dr. 2009) imamo ovakav pristup: "*The planar class of projections results from positioning the developable surface of a plane next to the reference globe and projecting the landmasses and graticule onto the plane.*" A u fusnoti: "*Some texts refer to the planar class of projections as azimuthal. We have chosen to use planar because it is easier to relate to the developable surface concept.*" Osim već navedene primjedbe u prethodnom odlomku, dodajmo da je upotreba ravnine kao razvojne plohe zbunjujuća jer razviti ravninu u nju samu sigurno nije naročito mudro!

U publikaciji USGS-a o kartografskim projekcijama razlikuju se azimutne projekcije po načinu projiciranja: geometrijski i matematički (USGS 2000). Kao da geometrija nije dio matematike! To je čini se uobičajen pristup u američkoj literaturi o kartografskim projekcijama, vidi npr. (Snyder 1987, Snyder i Voxland 1989).

Uspravna azimutna projekcija je preslikavanje zadano formulama

$$\rho = Rf(\varphi), \ \theta = \lambda - \lambda_0, \quad (35)$$

gdje su $\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\lambda \in [-\pi, \pi]$, konstante $R > 0$ i $\lambda_0 \in [-\pi, \pi]$, funkcija $\rho = Rf(\varphi)$ neprekidna, s pozitivnim vrijednostima i monotono padajuća ili monotono rastuća. Pri tome su ρ i θ koordinate točke u polarnom pravokutnom koordinatnom sustavu u ravnini. Uvjetom positive values and monotone decreasing or increasing. ρ and θ are coordinates of a point in the polar coordinate system in the plane. If the function $\rho = Rf(\varphi)$ decreases, then the image of the North Pole, or the centre of the circle that is the image of the North Pole, will be in the origin of the coordinate system in the plane of the projection. If $\rho = Rf(\varphi)$ increases, then the image of the South Pole, or the centre of the circle that is the image of the South Pole, will be in the origin of the coordinate system in the plane of the projection. For such mapping, according to (9)

$$E = \left(R\frac{df}{d\varphi}\right)^2, \; F = 0, \; G = R^2 f^2, \tag{36}$$

and the first differential form is

$$ds'^2 = \left(R\frac{df}{d\varphi}\right)^2 d\varphi^2 + R^2 f^2 d\lambda^2. \tag{37}$$

The local linear scale factor squared for mapping a sphere in normal aspect azimuthal projection (35) is

$$c^2 = \frac{\left(R\frac{df}{d\varphi}\right)^2 d\varphi^2 + R^2 f^2 d\lambda^2}{R^2 d\varphi^2 + R^2\cos^2\varphi d\lambda^2} = \frac{\left(\frac{df}{d\varphi}\right)^2 d\varphi^2 + f^2 d\lambda^2}{d\varphi^2 + \cos^2\varphi d\lambda^2}. \tag{38}$$

From (38) we can read the local linear scale factors along meridians and parallels as

$$h = h(\varphi) = \left|\frac{df}{d\varphi}\right|, \quad k = k(\varphi) = \frac{f(\varphi)}{\cos\varphi}. \tag{39}$$

Assuming that functions ρ, and then f, are monotone decreasing, then

$$\frac{df}{d\varphi} < 0, \text{ and } \quad h = h(\varphi) = -\frac{df}{d\varphi}. \tag{40}$$

If ρ, and then f, are monotone increasing functions, then

$$\frac{df}{d\varphi} > 0 \text{ and } \quad h = h(\varphi) = \frac{df}{d\varphi}. \tag{41}$$

4 Standard Parallels in Normal Aspect Azimuthal Projections

For normal aspect azimuthal projections, $F = 0$ and (39). Due to (32)–(34), along standard parallel $\varphi = \varphi_1$ it should be

$$h(\varphi_1) = k(\varphi_1) = 1, \tag{42}$$

or

$$\left|\frac{df}{d\varphi}(\varphi_1)\right| = 1 \tag{43}$$

and

$$\frac{f(\varphi_1)}{\cos\varphi_1} = 1, \text{ i.e. } f(\varphi_1) = \cos\varphi_1. \tag{44}$$

In this paper we will observe only azimuthal projections with at most one standard parallel, since it is obvious that the plane cannot cut the sphere into more than one parallel. However, for the sake of integrity, we should mention that there are also azimuthal projections with more than one standard parallel (Lapaine, 2015).

5 Secant Parallel in Normal Aspect Azimuthal Projection

Let us imagine a plane that cuts a sphere (1) along a parallel with latitude $\varphi_1 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. This is the secant parallel with the radius of that parallel equal to $R\cos\varphi_1$. The question arises: is this parallel also a standard parallel in a normal aspect azimuthal projection? The answer is provided by the relation

$$\rho(\varphi_1) = R\cos\varphi_1, \tag{45}$$

which is equivalent to (44). Accordingly, if (43) and (44) are valid for a normal aspect azimuthal projection, then the parallel with latitude φ_1 is simultaneously standard and secant. Let us examine the situation in different azimuthal projections.

6 Examples of Azimuthal Projections and Their Standard Parallels

6.1 Normal Aspect Azimuthal Projection Equidistant Along Meridians

Normal aspect azimuthal projection is equidistant along meridians if the local linear scale factor along meridians equals 1, i.e. if it is

$$h = h(\varphi) = \left|\frac{df}{d\varphi}\right| = 1. \tag{46}$$

Two cases should be distinguished. Let us assume that $\rho = Rf(\varphi)$ is a monotone decreasing function. We start with the differential equation

$$\frac{df}{d\varphi} = -1 \tag{47}$$

from which we obtain the solution by integration

$$f(\varphi) = -\varphi + C, \tag{48}$$

where C is a constant of integration. Taking into account that $\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and that the function $\rho = Rf(\varphi)$ should be positive, it follows that $C \geq \frac{\pi}{2}$.

pada funkcije $\rho = Rf(\varphi)$ postiže se da je slika Sjevernoga pola ili središte kružnice koja je slika Sjevernoga pola u ishodištu koordinatnog sustava u ravnini projekcije. Uvjetom rasta funkcije $\rho = Rf(\varphi)$ postiže se da je slika Južnoga pola ili središte kružnice koja je slika Južnoga pola u ishodištu koordinatnog sustava u ravnini projekcije. Za takvo preslikavanje je prema (9)

$$E = \left(R\frac{df}{d\varphi} \right)^2, \quad F = 0, \quad G = R^2 f^2, \tag{36}$$

i prva diferencijalna forma

$$ds'^2 = \left(R\frac{df}{d\varphi} \right)^2 d\varphi^2 + R^2 f^2 d\lambda^2. \tag{37}$$

Kvadrat faktora lokalnoga linearnog mjerila za preslikavanje sfere s pomoću uspravne azimutne projekcije (35) je

$$c^2 = \frac{\left(R\frac{df}{d\varphi} \right)^2 d\varphi^2 + R^2 f^2 d\lambda^2}{R^2 d\varphi^2 + R^2 \cos^2 \varphi d\lambda^2} = \frac{\left(\frac{df}{d\varphi} \right)^2 d\varphi^2 + f^2 d\lambda^2}{d\varphi^2 + \cos^2 \varphi d\lambda^2}. \tag{38}$$

Iz (38) možemo pročitati faktore lokalnih linearnih mjerila uzduž meridijana, odnosno paralela

$$h = h(\varphi) = \left| \frac{df}{d\varphi} \right|, \quad k = k(\varphi) = \frac{f(\varphi)}{\cos\varphi}. \tag{39}$$

Uz pretpostavku da su funkcije ρ, a onda i f, monotono padajuće vrijedi

$$\frac{df}{d\varphi} < 0, \text{ odnosno } h = h(\varphi) = -\frac{df}{d\varphi}. \tag{40}$$

Ako su funkcije ρ, a onda i f, monotono rastuće tada je

$$\frac{df}{d\varphi} > 0, \text{ odnosno } h = h(\varphi) = \frac{df}{d\varphi}. \tag{41}$$

4. Standardne paralele pri uspravnim azimutnim projekcijama

Za uspravne azimutne projekcije vrijedi $F = 0$ i (39). Uzduž standardne paralele $\varphi = \varphi_1$ treba zbog (32)–(34) biti

$$h(\varphi_1) = k(\varphi_1) = 1, \tag{42}$$

ili

$$\left| \frac{df}{d\varphi}(\varphi_1) \right| = 1 \tag{43}$$

i

$$\frac{f(\varphi_1)}{\cos\varphi_1} = 1, \text{ tj. } f(\varphi_1) = \cos\varphi_1. \tag{44}$$

U ovome radu promatrat ćemo samo azimutne projekcije s najviše jednom standardnom paralelom jer je očito da ravnina ne može sijeći sferu u više od jedne paralele. Ipak, radi cjelovitosti, spomenimo da postoje i azimutne projekcije s više od jedne standardne paralele (Lapaine, 2015).

5. Presječna paralela pri uspravnoj azimutnoj projekciji

Zamislimo ravninu koja siječe sferu uzduž paralele kojoj odgovara geografska širina $\varphi_1 \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right)$. To je presječna paralela čiji je polumjer na sferi jednak je $R\cos\varphi_1$. Postavlja se pitanje: je li ta paralela ujedno standardna paralela pri uspravnoj azimutnoj projekciji? Odgovor na postavljeno pitanje slijedi iz relacije

$$\rho(\varphi_1) = R\cos\varphi_1, \tag{45}$$

što je ekvivalentno relaciji (44). Prema tome, ako za uspravnu azimutnu projekciju vrijedi (43) i (44), paralela kojoj odgovara geografska širina φ_1 bit će istodobno i standardna i presječna. Pogledajmo sad kako je to kod raznih uspravnih azimutnih projekcija.

6. Primjeri uspravnih azimutnih projekcija i njihove standardne paralele

6.1. Uspravna azimutna projekcija ekvidistantna uzduž meridijana

Uspravna azimutna projekcija bit će ekvidistantna uzduž meridijana ako je faktor lokalnog linearnog mjerila uzduž meridijana jednak 1, tj. ako vrijedi

$$h = h(\varphi) = \left| \frac{df}{d\varphi} \right| = 1. \tag{46}$$

Razlikujemo dva slučaja. Uz pretpostavku da je funkcija monotono padajuća krećemo od diferencijalne jednadžbe

$$\frac{df}{d\varphi} = -1 \tag{47}$$

odakle se integracijom odmah dobije rješenje

$$f(\varphi) = -\varphi + C, \tag{48}$$

gdje je C konstanta integracije. Budući da je $\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$ i da funkcija $\rho = Rf(\varphi)$ mora biti pozitivna, to mora biti $C \geq \frac{\pi}{2}$. Dakle, jednadžbe uspravne azimutne projekcije ekvidistantne uzduž meridijana uz monotono padajuću

Thus, the equations of normal aspect azimuthal projection equidistant along meridians with a monotone decreasing function $\rho = Rf(\varphi)$ reads

$$\rho = R(C-\varphi)\ ,\ \theta = \lambda - \lambda_0, \tag{49}$$

where $\varphi \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, $\lambda \in [-\pi,\pi]$, the constants are $C \geq \frac{\pi}{2}$, $\lambda_0 \in [-\pi,\pi]$ and $R>0$ is the radius of the sphere. Let us suppose that $\varphi_1 \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ is the latitude of the standard parallel in this projection. Then, according to (43) and (44) for $f(\varphi) = C - \varphi$ it should follow that

$$\frac{df}{d\varphi}(\varphi_1) = -1 \tag{50}$$

which is clearly true for any $\varphi \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ and

$$f(\varphi_1) = C - \varphi_1 = \cos\varphi_1 \tag{51}$$

from which it immediately follows that

$$C = \varphi_1 + \cos\varphi_1\ . \tag{52}$$

The condition $C \geq \frac{\pi}{2}$, posed in the definition of normal aspect azimuthal projection equidistant along meridians in order to ensure $\rho = \rho(\varphi) \geq 0$ for any $\varphi \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, now gives

$$\varphi_1 + \cos\varphi_1 \geq \frac{\pi}{2} \tag{53}$$

which is fulfilled for $\varphi_1 = \frac{\pi}{2}$ only, and which is contrary to the assumption. So, there are no standard parallels except for the one that degenerates into the point for $C = \frac{\pi}{2}$ – the North Pole.

An analysis can be performed in the same way, assuming that the function $\rho = Rf(\varphi)$ is monotone increasing. In this case, the condition obtained is

$$-\varphi_1 + \cos\varphi_1 \geq \frac{\pi}{2} \tag{54}$$

fulfilled for $\varphi_1 = -\frac{\pi}{2}$ only, which is contrary to the assumption. That means that are no standard parallels, except for the one degenerating to the point when $C = \frac{\pi}{2}$ – the South Pole. Thus, *a map produced in normal aspect azimuthal projection equidistant along meridians cannot have a standard parallel. The North or South Poles are the only points that can be mapped without distortions.*

6.2 Normal Aspect Azimuthal Projection Equidistant along Parallels or Normal Aspect Orthographic Projection

Normal aspect azimuthal projection is equidistant along parallels, if the local linear scale factor along the parallels is equal to 1, i.e. if it is

$$k = k(\varphi) = \frac{f(\varphi)}{\cos\varphi} = 1\ . \tag{55}$$

Two cases should be distinguished. Let us suppose that $\rho = Rf(\varphi) = R\cos\varphi$ is monotone decreasing. Then, the equations of the normal aspect azimuthal projection equidistant along parallels are

$$\rho = R\cos\varphi\ ,\ \theta = \lambda - \lambda_0, \tag{56}$$

where $\varphi \in \left[0,\frac{\pi}{2}\right]$, $\lambda \in [-\pi,\pi]$, the constants are $\lambda_0 \in [-\pi,\pi]$ and $R>0$ is the radius of the sphere.

Let us assume that $\varphi_1 \in \left[0,\frac{\pi}{2}\right)$ is the latitude of the standard parallel in that projection. Then, according to (43) and (44) for $f(\varphi) = \cos\varphi$ it should be

$$\frac{df}{d\varphi}(\varphi_1) = -\sin\varphi_1 = -1 \tag{57}$$

and

$$f(\varphi_1) = \cos\varphi_1\ . \tag{58}$$

The relation (57) is valid for $\varphi_1 = \frac{\pi}{2}$ only, which is contrary to the assumption. So, there are no standard parallels except for the one that degenerated into the point – the North Pole.

An analysis can be performed in the same way, assuming that the function $\rho = Rf(\varphi)$ is monotone increasing. In this case, $\varphi \in \left[-\frac{\pi}{2},0\right]$ and $\varphi_1 \in \left(-\frac{\pi}{2},0\right]$ and the condition obtained is

$$\frac{df}{d\varphi}(\varphi_1) = -\sin\varphi_1 = 1 \tag{59}$$

and

$$f(\varphi_1) = \cos\varphi_1\ . \tag{60}$$

Relation (59) is true for $\varphi_1 = -\frac{\pi}{2}$ only, which is contrary to the assumption. Thus, *a map produced in normal aspect azimuthal projection equidistant along parallels cannot include a standard parallel. The North or South Poles are the only points that are mapped without distortions.*

funkciju $\rho = Rf(\varphi)$ su

$$\rho = R(C-\varphi)\,,\;\; \theta = \lambda - \lambda_0\,, \tag{49}$$

gdje su $\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\lambda \in [-\pi, \pi]$, konstante $\lambda_0 \in [-\pi, \pi]$ i $R>0$ polumjer sfere.

Pretpostavimo da je $\varphi_1 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ geografska širina standardne paralele u toj projekciji. Tada prema (43) i (44) za $f(\varphi) = C - \varphi$ mora biti

$$\frac{df}{d\varphi}(\varphi_1) = -1 \tag{50}$$

što očito vrijedi jer vrijedi za svaki $\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ i

$$f(\varphi_1) = C - \varphi_1 = \cos\varphi_1 \tag{51}$$

odakle neposredno slijedi

$$C = \varphi_1 + \cos\varphi_1\,. \tag{52}$$

Uvjet $C \geq \frac{\pi}{2}$ koji je postavljen u definiciji uspravne azimutne projekcije ekvidistantne uzduž meridijana kako bi bilo $\rho = \rho(\varphi) \geq 0$ za svaki $\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ sad prelazi u

$$\varphi_1 + \cos\varphi_1 \geq \frac{\pi}{2} \tag{53}$$

što je ispunjeno samo za $\varphi_1 = \frac{\pi}{2}$, a što je suprotno pretpostavci. Dakle, standardne paralele općenito nema, osim one koja je za $C = \frac{\pi}{2}$ degenerirala u točku – Sjeverni pol.

Na isti način može se provesti analiza uz pretpostavku da je funkcija $\rho = Rf(\varphi)$ monotono rastuća. U tom slučaju dobije se uvjet

$$-\varphi_1 + \cos\varphi_1 \geq \frac{\pi}{2} \tag{54}$$

što je ispunjeno samo za $\varphi_1 = -\frac{\pi}{2}$, što znači da standardne paralele općenito nema, osim one koja je za $C = \frac{\pi}{2}$ degenerirala u točku – Južni pol.

Prema tome, *karta izrađena u uspravnoj azimutnoj projekciji ekvidistantnoj uzduž meridijana ne može imati standardnu paralelu, Sjeverni ili Južni pol jedine su točke koje mogu biti bez deformacija.*

6.2. Uspravna azimutna projekcija ekvidistantna uzduž paralela ili uspravna ortografska projekcija

Uspravna azimutna projekcija bit će ekvidistantna uzduž paralela ako je faktor lokalnog linearnog mjerila uzduž paralela jednak 1, tj. ako vrijedi

$$k = k(\varphi) = \frac{f(\varphi)}{\cos\varphi} = 1\,. \tag{55}$$

Razlikujemo dva slučaja. Uz pretpostavku da je funkcija $\rho = Rf(\varphi) = R\cos\varphi$ monotono padajuća, dobivamo jednadžbe uspravne azimutne projekcije ekvidistantne uzduž paralela

$$\rho = R\cos\varphi,\quad \theta = \lambda - \lambda_0, \tag{56}$$

gdje su $\varphi \in \left[0, \frac{\pi}{2}\right]$, $\lambda \in [-\pi, \pi]$, konstante $\lambda_0 \in [-\pi, \pi]$ i $R>0$ polumjer sfere.

Pretpostavimo da je $\varphi_1 \in \left[0, \frac{\pi}{2}\right)$ geografska širina standardne paralele u toj projekciji. Tada prema (43) i (44) za $f(\varphi) = \cos\varphi$ mora biti

$$\frac{df}{d\varphi}(\varphi_1) = -\sin\varphi_1 = -1 \tag{57}$$

i

$$f(\varphi_1) = \cos\varphi_1\,. \tag{58}$$

Relacija (57) vrijedi samo za $\varphi_1 = \frac{\pi}{2}$ što je u suprotnosti s pretpostavkom. Dakle, standardne paralele općenito nema, osim one koja je degenerirala u točku – Sjeverni pol.

Na isti način može se provesti analiza uz pretpostavku da je funkcija $\rho = Rf(\varphi)$ monotono rastuća. U tom slučaju treba gledati $\varphi \in \left[-\frac{\pi}{2}, 0\right]$ i $\varphi_1 \in \left(-\frac{\pi}{2}, 0\right]$, a dobije se

$$\frac{df}{d\varphi}(\varphi_1) = -\sin\varphi_1 = 1 \tag{59}$$

i

$$f(\varphi_1) = \cos\varphi_1. \tag{60}$$

Relacija (59) vrijedi samo za $\varphi_1 = -\frac{\pi}{2}$ što je u suprotnosti s pretpostavkom. Prema tome, *karta izrađena u uspravnoj azimutnoj projekciji ekvidistantnoj uzduž paralela ne može imati standardnu paralelu. Sjeverni ili Južni pol jedine su točke koje se u toj projekciji preslikavaju bez deformacija.*

6.3 Normal Aspect Equal-Area Azimuthal Projection

Normal aspect azimuthal projection is equal-area if the product of local linear scale factors along meridians and parallels is equal to 1, i.e. if it is

$$kh = \frac{f(\varphi)}{\cos\varphi}\left|\frac{df}{d\varphi}\right| = 1 . \tag{61}$$

Two cases should be distinguished. Let us suppose that $\rho = Rf(\varphi)$ is a monotone decreasing function. We start with the differential equation

$$-\frac{f(\varphi)}{\cos\varphi}\frac{df}{d\varphi} = 1 \tag{62}$$

from which

$$fdf = -\cos\varphi d\varphi \tag{63}$$

and after integration, we obtain

$$f = \sqrt{2(C - \sin\varphi)} \ , \tag{64}$$

where C is a constant of integration. Since $\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and the expression under the root should be non-negative, it follows that $C \geq 1$. Thus, the equations of normal aspect equal-area azimuthal projection with the monotone decreasing function $\rho = Rf(\varphi)$ are

$$\rho = R\sqrt{2(C - \sin\varphi)} \ , \theta = \lambda - \lambda_0 , \tag{65}$$

where $\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\lambda \in [-\pi, \pi]$, the constants are $C \geq 1$, $\lambda_0 \in [-\pi, \pi]$ and the radius of the sphere is *R>0.*

If $\varphi_1 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is the latitude of the standard parallel in that projection, then according to (43) and (44) for $f(\varphi) = \sqrt{2(C - \sin\varphi)}$ it should be

$$\frac{df}{d\varphi}(\varphi_1) = \frac{-\cos\varphi_1}{\sqrt{2(C - \sin\varphi_1)}} = -1 \tag{66}$$

and

$$f(\varphi_1) = \sqrt{2(C - \sin\varphi_1)} = \cos\varphi_1 \ . \tag{67}$$

We see that both relations (66) and (67) are fulfilled for

$$C = \frac{1}{2}\cos^2\varphi_1 + \sin\varphi_1 . \tag{68}$$

The condition $C \geq 1$, given that $\rho = \rho(\varphi)$ is real for any $\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, is now

$$(\sin\varphi_1 - 1)^2 \leq 0 \ , \tag{69}$$

which is valid for $\sin\varphi_1 = 1$ only, i.e. for $\varphi_1 = \frac{\pi}{2}$, which is contrary to the assumption. So, there are no standard parallels except for one that degenerated into the point for $C = 1$ – the North Pole.

An analysis can be performed in the same way, assuming that the function $\rho = Rf(\varphi)$ is monotone increasing. In that case the condition obtained is

$$(\sin\varphi_1 + 1)^2 \leq 0 , \tag{70}$$

which is valid for $\sin\varphi_1 = -1$ only, i.e. for $\varphi_1 = -\frac{\pi}{2}$, which is contrary to the assumption. Thus, *a map produced in normal aspect equal-area azimuthal projection cannot include a standard parallel. The North or South Poles are the only points that can be mapped without distortions.*

6.4 Normal Aspect Conformal Azimuthal Projection or Normal Aspect Stereographic Projection

Normal aspect azimuthal projection is conformal if local linear scale factors along meridians and along parallels coincide, i.e. if it is

$$\left|\frac{df}{d\varphi}\right| = \frac{f(\varphi)}{\cos\varphi} . \tag{71}$$

Two cases should be distinguished. Let us suppose that $\rho = Rf(\varphi)$ is a monotone decreasing function. We start with the differential equation

$$-\frac{df}{d\varphi} = \frac{f(\varphi)}{\cos\varphi} \tag{72}$$

from which it follows

$$\frac{df}{f} = -\frac{d\varphi}{\cos\varphi} \tag{73}$$

and after integration

$$\ln f = -\ln\tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right) + \ln C \tag{74}$$

that is equivalent to

$$f = C\cot\left(\frac{\pi}{4} + \frac{\varphi}{2}\right) = C\tan\left(\frac{\pi}{4} - \frac{\varphi}{2}\right) , \tag{75}$$

where $C > 0$ is a constant of integration. Thus, the equations of normal aspect conformal azimuthal projection with the monotone decreasing function $\rho = Rf(\varphi)$ are

$$\rho = RC\tan\left(\frac{\pi}{4} - \frac{\varphi}{2}\right) , \ \theta = \lambda - \lambda_0 \tag{76}$$

6.3. Uspravna ekvivalentna azimutna projekcija

Uspravna azimutna projekcija bit će ekvivalentna ako je produkt faktora lokalnih linearnih mjerila uzduž meridijana i uzduž paralela jednak 1, tj. ako vrijedi

$$kh = \frac{f(\varphi)}{\cos\varphi}\left|\frac{df}{d\varphi}\right| = 1 \quad . \tag{61}$$

Razlikujemo dva slučaja. Uz pretpostavku da je funkcija $\rho = Rf(\varphi)$ monotono padajuća krećemo od diferencijalne jednadžbe

$$-\frac{f(\varphi)}{\cos\varphi}\frac{df}{d\varphi} = 1 \tag{62}$$

odakle je

$$fdf = -\cos\varphi d\varphi \tag{63}$$

i nakon integriranja

$$f = \sqrt{2(C - \sin\varphi)} \tag{64}$$

gdje je C konstanta integracije. Budući da je $\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ i da izraz ispod korijena treba biti nenegativan, mora biti $C \geq 1$. Dakle, jednadžbe uspravne ekvivalentne azimutne projekcije za monotono padajuće funkcije su

$$\rho = R\sqrt{2(C - \sin\varphi)}, \quad \theta = \lambda - \lambda_0 , \tag{65}$$

gdje su $\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\lambda \in [-\pi, \pi]$, konstante $C \geq 1$, $\lambda_0 \in [-\pi, \pi]$ i $R>0$ polumjer sfere.

Ako je $\varphi_1 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ geografska širina standardne paralele u toj projekciji, onda prema (43) i (44) za $f(\varphi) = \sqrt{2(C - \sin\varphi)}$ mora biti

$$\frac{df}{d\varphi}(\varphi_1) = \frac{-\cos\varphi_1}{\sqrt{2(C - \sin\varphi_1)}} = -1 \tag{66}$$

i

$$f(\varphi_1) = \sqrt{2(C - \sin\varphi_1)} = \cos\varphi_1. \tag{67}$$

Vidimo da su obje relacije (66) i (67) ispunjene za

$$C = \frac{1}{2}\cos^2\varphi_1 + \sin\varphi_1 . \tag{68}$$

Uvjet $C \geq 1$ da bi $\rho = \rho(\varphi)$ bilo realno za svaki $\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, sad prelazi u

$$(\sin\varphi_1 - 1)^2 \leq 0 , \tag{69}$$

što je ispunjeno samo za $\sin\varphi_1 = 1$, odnosno $\varphi_1 = \frac{\pi}{2}$ što je suprotno pretpostavci. Dakle, standardne paralele općenito nema, osim one koja je za $C = 1$ degenerirala u točku – Sjeverni pol.

Na isti način može se provesti analiza uz pretpostavku da je funkcija $\rho = Rf(\varphi)$ monotono rastuća. U tom slučaju dobije se uvjet

$$(\sin\varphi_1 + 1)^2 \leq 0 , \tag{70}$$

koji je ispunjen samo za $\sin\varphi_1 = -1$, odnosno $\varphi_1 = -\frac{\pi}{2}$ što je suprotno pretpostavci. Prema tome, *karta izrađena u uspravnoj ekvivalentnoj azimutnoj projekciji ne može imati standardnu paralelu. Sjeverni ili Južni pol jedine su točke koje mogu biti bez deformacija.*

6.4. Uspravna konformna azimutna projekcija ili uspravna stereografska projekcija

Uspravna azimutna projekcija bit će konformna ako su faktori lokalnih linearnih mjerila uzduž meridijana i uzduž paralela međusobno jednaki, tj. ako vrijedi

$$\left|\frac{df}{d\varphi}\right| = \frac{f(\varphi)}{\cos\varphi} \quad . \tag{71}$$

Razlikujemo dva slučaja. Uz pretpostavku da je funkcija $\rho = Rf(\varphi)$ monotono padajuća krećemo od diferencijalne jednadžbe

$$-\frac{df}{d\varphi} = \frac{f(\varphi)}{\cos\varphi} \tag{72}$$

odakle slijedi

$$\frac{df}{f} = -\frac{d\varphi}{\cos\varphi} \tag{73}$$

i nakon integriranja najprije

$$\ln f = -\ln\tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right) + \ln C \tag{74}$$

i zatim

$$f = C\cot\left(\frac{\pi}{4} + \frac{\varphi}{2}\right) = C\tan\left(\frac{\pi}{4} - \frac{\varphi}{2}\right), \tag{75}$$

gdje je $C > 0$ konstanta integracije. Dakle, jednadžbe uspravne konformne azimutne projekcije za koju je funkcija $\rho = Rf(\varphi)$ monotono padajuća su

$$\rho = RC\tan\left(\frac{\pi}{4} - \frac{\varphi}{2}\right), \theta = \lambda - \lambda_0 \tag{76}$$

where $\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\lambda \in [-\pi, \pi]$, the constants are $C > 0$, $\lambda_0 \in [-\pi, \pi]$ and the radius of the sphere is $R>0$.

If $\varphi_1 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is the latitude of a standard parallel in that projection, then according to (43) and (44) for

$$f(\varphi) = C\tan\left(\frac{\pi}{4} - \frac{\varphi}{2}\right) \qquad \text{it should be}$$

$$\frac{df}{d\varphi}(\varphi_1) = -\frac{C}{2\cos^2\left(\frac{\pi}{4} - \frac{\varphi_1}{2}\right)} = = -\frac{C}{1+\cos\left(\frac{\pi}{2} - \varphi_1\right)} = -\frac{C}{1+\sin\varphi_1} = -1 \qquad (77)$$

and

$$f(\varphi_1) = C\tan\left(\frac{\pi}{4} - \frac{\varphi_1}{2}\right) = \frac{C\cos\varphi_1}{1+\sin\varphi_1} = \cos\varphi_1 \,. \quad (78)$$

We see that both relations (77) and (78) are fulfilled for

$$C = 1 + \sin\varphi_1 \,. \qquad (79)$$

The condition $C > 0$ is now $\sin\varphi_1 > -1$, i.e. φ_1 can have any value from the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

An analysis can be performed in the same way, assuming that the function $\rho = Rf(\varphi)$ is monotone increasing. In that case, the condition obtained is $\sin\varphi_1 < 1$, i.e. φ_1 can have any value from the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

To conclude, *a map produced in normal aspect conformal azimuthal or stereographic projection, and set parallel to the equatorial plane at the height* $R\sin\varphi_1$ *above or below the equatorial plane and with its centre on the Z axis, will cut the sphere along a standard parallel of latitude* φ_1 *(in the circle with the radius* $R\cos\varphi_1$*) if, and only if, the formula of the radius of the parallels is*

$$\rho = R(1+\sin\varphi_1)\tan\left(\frac{\pi}{4} - \frac{\varphi}{2}\right) = R(1+\sin\varphi_1)\frac{\cos\varphi}{1+\sin\varphi}$$

or

$$\rho = R(1-\sin\varphi_1)\tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right) = R(1-\sin\varphi_1)\frac{\cos\varphi}{1-\sin\varphi} \,.$$

Consequently, no normal aspect stereographic projection shows that the standard and secant parallels coincide in general. This property is only true for normal aspect stereographic projections with the function $\rho = Rf(\varphi)$ monotone decreasing and the constant C in the form $C = 1 + \sin\varphi_1$, and normal aspect stereographic projections with the function $\rho = Rf(\varphi)$ monotone increasing and the constant C in the form $C = 1 - \sin\varphi_1$ where φ_1 is the latitude of the standard and secant parallels simultaneously.

Finally, based on the previous derivations, it can be concluded that if $0 < C \le 2$, then a normal aspect stereographic projection where the standard and secant parallels coincide can exist, and for $C > 2$, cannot exist. If $C \le 0$, then the stereographic projection is not defined.

7 Conclusion

- Some normal aspect azimuthal projections cannot have any standard parallel. Examples of such projections are normal aspect azimuthal equidistant along meridians, normal aspect azimuthal equidistant along parallels (orthographic) and normal aspect equal-area azimuthal projections. Those projections can have only one point without distortion – the image of the North or South Pole.
- The normal aspect conformal (stereographic) azimuthal projection may have a standard parallel, but this depends on the choice of factor in the formula for the parallel radius $\rho = \rho(\varphi)$. When this factor is selected appropriately, a map made in the stereographic projection and placed parallel to the equatorial plane at the height $R\sin\varphi_1$ above or below the equatorial plane will cut the sphere along a standard parallel of latitude φ_1 in the circle/parallel with the radius $R\cos\varphi_1$. In this case, the standard parallel and the secant parallel coincide.
- In the literature on map projections, some formulae for azimuthal projections are usually provided that include only a monotone increasing function $\rho = \rho(\varphi)$, but without explicitly stating this. This paper covers both cases: the function $\rho = \rho(\varphi)$ can be monotone decreasing or increasing. It will be noticed that the formulae for maps with the monotone increasing $\rho = \rho(\varphi)$ function can be obtained from the corresponding formulas with monotone decreasing function $\rho = \rho(\varphi)$ by the formal substitution of φ for $-\varphi$. Of course, the reverse is also true.

Acknowledgements

I am grateful to Lynne E. Usery for helping me access the article on map projections from the National Atlas of the United States of America, which was withdrawn from the Internet at the end of 2014.

I also extend my thanks to the reviewers for their helpful advice.

gdje je $\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\lambda \in [-\pi, \pi]$, konstante $C > 0$, $\lambda_0 \in [-\pi, \pi]$ i R>0 polumjer sfere.

Ako je $\varphi_1 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ geografska širina standardne paralele u toj projekciji onda prema (43) i (44) za

$$f(\varphi) = C \tan\left(\frac{\pi}{4} - \frac{\varphi}{2}\right) \quad \text{mora biti}$$

$$\frac{df}{d\varphi}(\varphi_1) = -\frac{C}{2\cos^2\left(\frac{\pi}{4} - \frac{\varphi_1}{2}\right)} = \tag{77}$$

$$= -\frac{C}{1 + \cos\left(\frac{\pi}{2} - \varphi_1\right)} = -\frac{C}{1 + \sin\varphi_1} = -1$$

i

$$f(\varphi_1) = C \tan\left(\frac{\pi}{4} - \frac{\varphi_1}{2}\right) = \frac{C\cos\varphi_1}{1 + \sin\varphi_1} = \cos\varphi_1 . \tag{78}$$

Vidimo da su obje relacije (77) i (78) ispunjene za

$$C = 1 + \sin\varphi_1 . \tag{79}$$

Uvjet $C > 0$, sad prelazi u $\sin\varphi_1 > -1$, tj. φ_1 može imati bilo koju vrijednost iz intervala $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Na isti način može se provesti analiza uz pretpostavku da je funkcija $\rho = Rf(\varphi)$ monotono rastuća. U tom slučaju dobije se uvjet $\sin\varphi_1 < 1$, tj. φ_1 može imati bilo koju vrijednost iz intervala $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Prema tome, *karta izrađena u uspravnoj azimutnoj konformnoj, tj. uspravnoj stereografskoj projekciji i postavljena paralelno s ravninom ekvatora na visini* $R\sin\varphi_1$ *iznad, odnosno ispod ravnine ekvatora i tako da joj je središte na osi Z, siječi će sferu uzduž standardne paralele kojoj odgovara geografska širina* φ_1 *(u kružnici polumjera* $R\cos\varphi_1$*) ako i samo ako formula te projekcije glasi:*

$$\rho = R(1 + \sin\varphi_1)\tan\left(\frac{\pi}{4} - \frac{\varphi}{2}\right) = R(1 + \sin\varphi_1)\frac{\cos\varphi}{1 + \sin\varphi}$$

ili

$$\rho = R(1 - \sin\varphi_1)\tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right) = R(1 - \sin\varphi_1)\frac{\cos\varphi}{1 - \sin\varphi} .$$

Prema tome, nema svaka uspravna stereografska projekcija svojstvo da se standardna i presječna paralela poklapaju. To svojstvo imaju samo one uspravne stereografske projekcije kod kojih je funkcija $\rho = Rf(\varphi)$ monotono padajuća i kod kojih je konstanta C oblika $C = 1 + \sin\varphi_1$ te one uspravne stereografske projekcije kod kojih je funkcija $\rho = Rf(\varphi)$ monotono rastuća i kod kojih je konstanta C oblika $C = 1 - \sin\varphi_1$, gdje je φ_1 geografska širina standardne i presječne paralele.

Konačno, na temelju prethodnih izvoda može se zaključiti da ako je $0 < C \leq 2$ onda postoji uspravna stereografska projekcija sa svojstvom da se standardna i sjekuća paralela poklapaju, a za $C > 2$ ne postoji. Za $C \leq 0$ uspravna stereografska projekcija nije definirana.

7. Zaključak

- Neke uspravne azimutne projekcije ne mogu imati ni jednu standardnu paralelu. Primjeri takvih projekcija su uspravna azimutna ekvidistantna uzduž meridijana, uspravna azimutna ekvidistantna uzduž paralela (ortografska) i uspravna ekvivalentna azimutna projekcija. Te projekcije mogu imati samo jednu točku bez deformacija – sliku Sjevernog, odnosno Južnog pola.
- Uspravna konformna (stereografska) azimutna projekcija može imati standardnu paralelu, no to ovisi o izboru faktora u formuli za polumjer paralela $\rho = \rho(\varphi)$. Uz odgovarajući izbor tog faktora karta izrađena u stereografskoj projekciji i postavljena paralelno s ravninom ekvatora na visini $R\sin\varphi_1$ iznad, odnosno ispod ravnine ekvatora, siječi će sferu uzduž standardne paralele kojoj odgovara geografska širina φ_1 u kružnici/paraleli polumjera $R\cos\varphi_1$. U tom se slučaju standardna i presječna paralela poklapaju.
- U literaturi o kartografskim projekcijama obično se daju formule za uspravne azimutne projekcije kod kojih je funkcija $\rho = \rho(\varphi)$ monotono padajuća, ali se to svojstvo eksplicitno ne navodi. U ovom radu obuhvaćena su oba slučaja: funkcija $\rho = \rho(\varphi)$ može biti monotono padajuća ili monotono rastuća. Može se uočiti da se formule za karte izrađene uz pretpostavku da je funkcija $\rho = \rho(\varphi)$ monotono rastuća mogu dobiti iz formula za karte izrađene uz pretpostavku da je funkcija $\rho = \rho(\varphi)$ monotono padajuća formalnom zamjenom geografske širine φ s $-\varphi$. Naravno, vrijedi i obratno.

Zahvala

Zahvaljujem Lynnu E. Useryju koji mi je pomogao u dostupnosti članka o kartografskim projekcijama iz The National Atlas of the United States of America, a koji je povučen s interneta krajem 2014. godine.

Zahvaljujem recenzentima na korisnim savjetima.

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