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Distortions in Affine Mapping

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Abstract. Affine mappings have a long tradition and are applied in various disciplines. Distortions of affine mappings between planes are discussed in this paper using the methodology taken from the theory of map projections. Formulae for the linear scale are derived, and it is shown that distances are generally not preserved. The length distortion depends on the direction. The distribution of the linear distortions around a point can be visualized by using the curve of local linear distortions or the distortion ellipse. Next, changes in angles and areas are explained. The paper concludes with an interpretation of the distortion ellipse as the image of a unit circle.

Keywords: affine mapping, mapping distortions, length distortion, angle distortion, area distortion, curve of linear local distortion, distortion ellipse

1 Introduction

Affine transformation of coordinates is used in different fields of surveying and geodesy (Lapaine, Frančula 1994), but in general, it is not easy to find a satisfactory approach to its distortions. Therefore, this paper will give a detailed elaboration of the distortions inherent in any affine mapping of a plane onto a plane. Distortions of length, area and angle will be considered using the approach familiar in the theory of map projections.

2 Distortions in Affine Mapping of a Plane onto a Plane

It is known that equations of affine mapping of a plane onto a plane can be expressed in the following way:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \tag{1}$$

where x, y and x', y' are coordinates of a point and its image. The point with coordinates x, y is given in a coordinate system xOy , and its image with coordinates x', y' in another plane coordinate system $x'O'y'$. The real numbers $a_1, b_1, c_1, a_2, b_2, c_2$ are the so-called parameters of affine mapping. The matrix

$$M = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

is matrix of the affine mapping. In this paper we will suppose that matrix M is regular one, i.e.

$$\det M = a_1 b_2 - a_2 b_1 \neq 0.$$

From (1) it is easy to obtain the relations between differentials

$$\begin{bmatrix} dx' \\ dy' \end{bmatrix} = M \begin{bmatrix} dx \\ dy \end{bmatrix}. \tag{2}$$

2.1 Linear scale in affine mapping

If we denote

$$Q = M^T M = \begin{bmatrix} e & f \\ f & g \end{bmatrix}, \tag{3}$$

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Deformacije pri afinom preslikavanju

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Sažetak. Afino preslikavanje ima dugogodišnju tradiciju i primjenu u različitim područjima. U ovome radu istražuju se deformacije koje nastaju pri afinom preslikavanju ravnine na ravninu. Pri tome se koristi metodologija poznata u teoriji kartografskih projekcija. Izvode se odgovarajuće formule za linearno mjerilo na temelju koji se pokazuje se da se pri takvom preslikavanju općenito ne čuvaju udaljenosti. Deformacije duljina ovise o smjeru, a raspodjela deformacija duljina oko pojedine točke može se opisati s pomoću krivulje lokalnih deformacija duljina ili s pomoću elipse deformacija. Nadalje, utvrđuje se na koji način se mijenjaju kutovi i površine pri afinom preslikavanju ravnine na ravninu. Na kraju se daje interpretacija elipse deformacija kao afine slike jedinične kružnice.

Ključne riječi: afino preslikavanje, deformacije preslikavanja, deformacije duljina, deformacije kutova, deformacije površina, krivulja lokalnih deformacija duljina, elipsa deformacija

1. Uvod

Afina transformacija koordinata ravnine na ravninu upotrebljava se ili se upotrebljavala u raznim područjima geodezije (Lapaine i Frančula 1994), ali se općenito malo govori o deformacijama koje neminovno sa sobom nosi svako afino preslikavanje. Stoga će se u ovome radu detaljno raspraviti o deformacijama koje se uobičajeno promatraju u teoriji kartografskih projekcija, a to su deformacije duljina, površina i kutova.

2. Deformacije pri afinom preslikavanju ravnine na ravninu

Poznato je da se jednadžbe afinog preslikavanja ravnine na ravninu mogu napisati u obliku

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \tag{1}$$

gdje su x, y i x', y' koordinate točke i njezine slike. Točka s koordinatama x, y je u sustavu xOy, a njezina slika s koordinatama x', y' u koordinatnom sustavu $x'O'y'$. Brojevi a_1, b_1, c_1, a_2, b_2, c_2 su parametri afinog preslikavanja. Matrica

$$M = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

je matrica afinog preslikavanja. U ovome radu pretpostavit ćemo da je matrica M regularna, tj. da vrijedi

$$\det M = a_1 b_2 - a_2 b_1 \neq 0.$$

Iz izraza (1) lako se dobiju veze među diferencijalima

$$\begin{bmatrix} dx' \\ dy' \end{bmatrix} = M \begin{bmatrix} dx \\ dy \end{bmatrix}. \tag{2}$$

2.1. Linearno mjerilo pri afinom preslikavanju

Označimo

$$Q = M^T M = \begin{bmatrix} e & f \\ f & g \end{bmatrix}, \tag{3}$$

then it follows that

$$e = a_1^2 + a_2^2, \quad f = a_1 b_1 + a_2 b_2, \quad g = b_1^2 + b_2^2. \tag{4}$$

Furthermore, the differentials of the length of an arc and its image will be

$$ds^2 = \begin{bmatrix} dx \\ dy \end{bmatrix}^T \begin{bmatrix} dx \\ dy \end{bmatrix} = dx^2 + dy^2, \tag{5}$$

$$ds'^2 = dx'^2 + dy'^2 = \begin{bmatrix} dx' \\ dy' \end{bmatrix}^T \begin{bmatrix} dx' \\ dy' \end{bmatrix} =$$

$$= \begin{bmatrix} dx \\ dy \end{bmatrix}^T Q \begin{bmatrix} dx \\ dy \end{bmatrix} = edx^2 + 2fdxdy + gdy^2, \tag{6}$$

which means that for the linear scale c of affine mapping at any point and in any direction,

$$c^2 = \frac{ds'^2}{ds^2} = \frac{\begin{bmatrix} dx \\ dy \end{bmatrix}^T Q \begin{bmatrix} dx \\ dy \end{bmatrix}}{\begin{bmatrix} dx \\ dy \end{bmatrix}^T \begin{bmatrix} dx \\ dy \end{bmatrix}} = \frac{edx^2 + 2fdxdy + gdy^2}{dx^2 + dy^2}. \tag{7}$$

From the last relation (7) it follows immediately that affine mapping generally does not preserve distances.

If Q is the identity matrix, then $c^2 = 1$, $e = g = 1$, $f = 0$, we have an isometry, and matrix M should be orthogonal. In that case all distances, anlges and areas are preserved.

If Q is proportional to the identity matrix, then $c^2 = k$, $e = g = 1$, $f = 0$, where k is a coefficient of proportionality, we have a similarity, and matrix M should be a matrix proportional to the orthogonal one. In that case all distances are stretched in the same proportion and angles are preserved. That is why similarity is often known as conformal mapping. Isometry is a special case of similarity when $c^2 = k = 1$.

Although the Gauss coefficients e, f and g are constants of affine mapping, the linear scale depends on directions at any point. For example, the linear scale along the x axis is determined by $dy = 0$ and amounts to

$$m = \sqrt{e} = \sqrt{a_1^2 + a_2^2}. \tag{8}$$

The linear scale along the y axis is determined by $dx = 0$ and amounts to

$$n = \sqrt{g} = \sqrt{b_1^2 + b_2^2}. \tag{9}$$

This relation (6) enables interpretation of the linear scale distribution around a point. If we apply the substitution

$$\cos\alpha = \frac{dx}{ds}, \sin\alpha = \frac{dy}{ds} \tag{10}$$

then (6) can be transformed into

$$c^2(\alpha) = \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}^T Q \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix} = e\cos^2\alpha + f\sin 2\alpha + g\sin^2\alpha. \tag{11}$$

Although the expression (11) is relatively simple, it is not easy to draw conclusions directly from it about the function $c = c(\alpha)$, i.e. about the behaviour of linear scale around the point in question. This is why we will proceed as follows. Each real symmetric matrix Q can be factorized thus

$$Q = V \Lambda V^T, \tag{12}$$

where V is an orthogonal matrix, and Λ a diagonal matrix. The columns of matrix V are orthonormal eigenvectors of matrix Q, while diagonal elements of the matrix Λ are corresponding eigenvalues. It is the so-called *eigenvalue decomposition*, which is a special case of *singular value decomposition* – SVD. If we denote

$$V = \begin{bmatrix} \cos\theta_1^Q & -\sin\theta_1^Q \\ \sin\theta_1^Q & \cos\theta_1^Q \end{bmatrix} \text{ i } \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \tag{13}$$

then (12) is equivalent to

$$\begin{bmatrix} e & f \\ f & g \end{bmatrix} = \begin{bmatrix} \lambda_1\cos^2\theta_1^Q + \lambda_2\sin^2\theta_1^Q & (\lambda_1 - \lambda_2)\sin\theta_1^Q\cos\theta_1^Q \\ (\lambda_1 - \lambda_2)\sin\theta_1^Q\cos\theta_1^Q & \lambda_1\sin^2\theta_1^Q + \lambda_2\cos^2\theta_1^Q \end{bmatrix}. \tag{14}$$

The expression (14) is equivalent to

$$\begin{aligned} \lambda_1 \cos^2\theta_1^Q + \lambda_2 \sin^2\theta_1^Q &= e \\ \lambda_1 \sin^2\theta_1^Q + \lambda_2 \cos^2\theta_1^Q &= g \\ (\lambda_1 - \lambda_2)\sin\theta_1^Q \cos\theta_1^Q &= f. \end{aligned} \tag{15}$$

from which is it easy to derive

$$e + g = \lambda_1 + \lambda_2 \text{ i } eg - f^2 = \lambda_1 \lambda_2. \tag{16}$$

By using Viète's formulae, it can be concluded that λ_1, λ_2 are roots of the quadratic equation

$$\lambda^2 - (e+g)\lambda + (eg - f^2) = 0, \tag{17}$$

or

$$\lambda_{1,2} = \frac{1}{2}(e + g \pm K), \tag{18}$$

where

Tada je

$$e = a_1^2 + a_2^2, \quad f = a_1 b_1 + a_2 b_2, \quad g = b_1^2 + b_2^2. \tag{4}$$

Nadalje, diferencijali duljine luka krivulje i njezine slike bit će

$$ds^2 = \begin{bmatrix} dx \\ dy \end{bmatrix}^T \begin{bmatrix} dx \\ dy \end{bmatrix} = dx^2 + dy^2, \tag{5}$$

$$ds'^2 = dx'^2 + dy'^2 = \begin{bmatrix} dx' \\ dy' \end{bmatrix}^T \begin{bmatrix} dx' \\ dy' \end{bmatrix} =$$

$$= \begin{bmatrix} dx \\ dy \end{bmatrix}^T Q \begin{bmatrix} dx \\ dy \end{bmatrix} = edx^2 + 2fdxdy + gdy^2, \tag{6}$$

pa za linearno mjerilo c afinog preslikavanja u bilo kojoj točki i u bilo kojem smjeru vrijedi

$$c^2 = \frac{ds'^2}{ds^2} = \frac{\begin{bmatrix} dx \\ dy \end{bmatrix}^T Q \begin{bmatrix} dx \\ dy \end{bmatrix}}{\begin{bmatrix} dx \\ dy \end{bmatrix}^T \begin{bmatrix} dx \\ dy \end{bmatrix}} = \frac{edx^2 + 2fdxdy + gdy^2}{dx^2 + dy^2}. \tag{7}$$

Dakle, afino preslikavanje općenito ne čuva udaljenosti.

Ako je Q jedinična matrica tada je $c^2 = 1$, riječ je o izometriji i matrica M je ortogonalna. U tom slučaju sve udaljenosti, svi kutovi i sve površine su sačuvane.

Ako je Q proporcionalna jediničnoj matrici tada je $c^2 = k$, gdje je k koeficijent proporcionalnosti, riječ je o sličnosti, a matrica M mora biti proporcionalna ortogonalnoj matrici. U tom slučaju sve duljine su rastegnute u istom omjeru i kutovi se sačuvani. Zbog toga se sličnost često naziva konformnim preslikavanjem. Izometrija je posebna slučaj sličnosti kad je $c^2 = k = 1$.

Iako su Gaussovi koeficijenti e, f i g konstante afinog preslikavanja, linearno mjerilo u svakoj točki ovisi o smjeru. Npr. mjerilo uzduž osi x dobije se za $dy = 0$ i iznosi

$$m = \sqrt{e} = \sqrt{a_1^2 + a_2^2}. \tag{8}$$

Mjerilo uzduž osi y dobije se za $dx = 0$ i iznosi

$$n = \sqrt{g} = \sqrt{b_1^2 + b_2^2}. \tag{9}$$

Iz formule (6) može se doći do interpretacije raziobe linearnog mjerila oko neke točke. Ako uvedemo supstituciju

$$\cos\alpha = \frac{dx}{ds}, \sin\alpha = \frac{dy}{ds} \tag{10}$$

možemo (6) dobiti u obliku

$$c^2(\alpha) = \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}^T Q \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix} = e\cos^2\alpha + f\sin 2\alpha + g\sin^2\alpha. \tag{11}$$

Premda je izraz (11) relativno jednostavan, iz njega nije lako izravno izvesti zaključak o funkciji $c = c(\alpha)$, tj. o ponašanju linearnog mjerila oko promatrane točke. Stoga ćemo postupiti na sljedeći način. Svaka realna simetrična matrica Q može se rastaviti na ovakav produkt

$$Q = V \Lambda V^T, \tag{12}$$

gdje je V ortogonalna matrica, a Λ dijagonalna matrica. Pri tome su stupci matrice V ortonormirani svojstveni vektori matrice Q, a dijagonalni elementi matrice Λ odgovarajuće svojstvene vrijednosti. To je tzv. rastav po svojstvenim vrijednostima što je poseban slučaj rastava po singularnim vrijednostima (*Singular Value Decomposition* – SVD). Označimo li

$$V = \begin{bmatrix} \cos\theta_1^Q & -\sin\theta_1^Q \\ \sin\theta_1^Q & \cos\theta_1^Q \end{bmatrix} \text{ i } \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \tag{13}$$

onda je (12) ekvivalentno s

$$\begin{bmatrix} e & f \\ f & g \end{bmatrix} = \begin{bmatrix} \lambda_1\cos^2\theta_1^Q + \lambda_2\sin^2\theta_1^Q & (\lambda_1 - \lambda_2)\sin\theta_1^Q\cos\theta_1^Q \\ (\lambda_1 - \lambda_2)\sin\theta_1^Q\cos\theta_1^Q & \lambda_1\sin^2\theta_1^Q + \lambda_2\cos^2\theta_1^Q \end{bmatrix}. \tag{14}$$

Izraz (14) je ekvivalentan s

$$\begin{aligned} \lambda_1 \cos^2\theta_1^Q + \lambda_2 \sin^2\theta_1^Q &= e \\ \lambda_1 \sin^2\theta_1^Q + \lambda_2 \cos^2\theta_1^Q &= g \\ (\lambda_1 - \lambda_2)\sin\theta_1^Q\cos\theta_1^Q &= f. \end{aligned} \tag{15}$$

Iz (15) možemo lako dobiti

$$e + g = \lambda_1 + \lambda_2 \text{ i } eg - f^2 = \lambda_1\lambda_2. \tag{16}$$

Primjenom Vièteovih formula možemo zaključiti da su λ_1 i λ_2 rješenja kvadratne jednažbe

$$\lambda^2 - (e+g)\lambda + (eg - f^2) = 0, \tag{17}$$

tj.

$$\lambda_{1,2} = \frac{1}{2}(e + g \pm K), \tag{18}$$

gdje smo s K označili

$$K = \sqrt{(e+g)^2 - 4(eg - f^2)} = \sqrt{(e-g)^2 + 4f^2}. \tag{19}$$

Nije teško vidjeti da će obje svojstvene vrijednosti λ_1 i λ_2 biti pozitivne. Ako je $\lambda_1 = \lambda_2$ tada je $K = 0$, tj. $e = g$ i $f = 0$.

$$K=\sqrt{(e+g)^2-4(eg-f^2)}=\sqrt{(e-g)^2+4f^2}. \quad (19)$$

It is easy to see that both eigenvalues λ1, λ2 are positive numbers.

If $\lambda_1=\lambda_2$ then $K=0$, i.e. $e=g$ and $f=0$. That means the mapping is a similarity.

If we suppose that $\lambda_1>\lambda_2$, then it follows that

$$\lambda_1-\lambda_2=K$$

and from the last equation in (15), it follows that

$$\sin 2\theta_1^Q=\frac{2f}{K},\ \cos 2\theta_1^Q=\frac{e-g}{K},\ \tan 2\theta_1^Q=\frac{2f}{e-g} \quad (20)$$

and then

$$\tan\theta_1^Q=\frac{\lambda_1-e}{f}=\frac{f}{e-\lambda_2}=\frac{1}{2f}(g-e+K). \quad (21)$$

Factorization (12) enables us to write (11) in the following form

$$c^2(\alpha)=\begin{bmatrix}\cos\alpha\\ \sin\alpha\end{bmatrix}^T\begin{bmatrix}\cos\theta_1^Q & -\sin\theta_1^Q\\ \sin\theta_1^Q & \cos\theta_1^Q\end{bmatrix}\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}\begin{bmatrix}\cos\theta_1^Q & \sin\theta_1^Q\\ -\sin\theta_1^Q & \cos\theta_1^Q\end{bmatrix}\begin{bmatrix}\cos\alpha\\ \sin\alpha\end{bmatrix}=$$

$$=\begin{bmatrix}\cos(\alpha-\theta_1^Q)\\ \sin(\alpha-\theta_1^Q)\end{bmatrix}^T\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}\begin{bmatrix}\cos(\alpha-\theta_1^Q)\\ \sin(\alpha-\theta_1^Q)\end{bmatrix}. \quad (22)$$

By using substitution

$$t=\alpha-\theta_1^Q \quad (23)$$

the expression (22) proceeds to

$$c^2(t)=\begin{bmatrix}\cos t\\ \sin t\end{bmatrix}^T\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}\begin{bmatrix}\cos t\\ \sin t\end{bmatrix}=$$

$$=\lambda_1\cos^2 t+\lambda_2\sin^2 t, \quad (24)$$

or to

$$c^2(t)=\frac{1}{2}(e+g+K\cos 2t). \quad (25)$$

Since the cosine function has a maximum value of 1, and a minimum value of –1, it is easy to draw conclusions from the relation (25) on the maximum and minimum of the linear scale $c=c(t)$. If the maximum linear scale is A, and the minimum B, then

$$A^2=\max c^2=\lambda_1=\frac{1}{2}(e+g+K) \text{ for } t=k\pi,\ k\in Z,$$

$$\text{i.e. } \alpha=\theta_1^Q+k\pi,\ k\in Z$$

$$B^2=\min c^2=\lambda_2=\frac{1}{2}(e+g-K) \text{ for } t=\frac{\pi}{2}+k\pi,\ k\in Z,$$

$$\text{i.e. } \alpha=\theta_1^Q+\frac{\pi}{2}+k\pi,\ k\in Z. \quad (26)$$

The relation (21) determines the direction of the maximum linear scale at the point of interest. From the relation (26) it follows that the directions of the minimum linear scale and maximal linear scale form a right angle. In fact, this is the consequence of the theorem that eigenvectors of a matrix are mutually orthogonal. The directions that correspond to the maximum and minimum linear scales are known as main distortion directions.

If the function $c=c(\alpha)$ from (11), or $c=c(t)$ from (24) or (25) is represented in graphic form (Figure 1), then a curve is obtained, known in the theory of errors as a curve of errors, and in the theory of map projections as a curve of locale linear distortions (Greek ποδηρης, French *podaire*, Lapaine 1989, 1991). From the derivation given above we can conclude that the curve of linear distortions in affine mapping can be visualized by using an equation in the polar coordinate system

$$c(t)=\sqrt{A^2\cos^2 t+B^2\sin^2 t} \quad (27)$$

or in the Cartesian coordinate system x, y by using its equation in parametric form

$$\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}x(t)\\ y(t)\end{bmatrix}=c(t)\begin{bmatrix}\cos t\\ \sin t\end{bmatrix}=\sqrt{A^2\cos^2 t+B^2\sin^2 t}\begin{bmatrix}\cos t\\ \sin t\end{bmatrix} \quad (28)$$

for $t\in[0,2\pi]$, taking into account the angle θ_1^Q, because t is defined by the relation (23), while the angle θ_1^Q can be calculated by using (21). Since A and B do not depend on the point in question, it can be concluded that the curve of local linear distortion does not depend on the point either. The curve showing the distribution of linear distortion in the original plane is the same at any point.

If we eliminate the parameter t from the equation (28) we can get the equation of the curve of linear distortions in its implicit form

$$(x^2+y^2)^2=A^2x^2+B^2y^2$$

from which we can see that the curve is of 4th order. If we want to visualize the curve of local linear distortions with its centre at the point with coordinates x_0, y_0, without taking into account the main axes directions, then we can use these parametric equations

Nadalje, ako uzmemo da je $\lambda_1 > \lambda_2$, onda je

$$\lambda_1 - \lambda_2 = K$$

pa iz posljednje jednadžbe u (15) slijedi

$$\sin 2\theta_1^Q = \frac{2f}{K}, \cos 2\theta_1^Q = \frac{e-g}{K}, \tan 2\theta_1^Q = \frac{2f}{e-g} \tag{20}$$

i zatim

$$\tan\theta_1^Q = \frac{\lambda_1 - e}{f} = \frac{f}{e-\lambda_2} = \frac{1}{2f}(g-e+K). \tag{21}$$

Rastav (12) omogućava da se (11) napiše u obliku

$$c^2(\alpha) = \begin{bmatrix}\cos\alpha\\ \sin\alpha\end{bmatrix}^T \begin{bmatrix}\cos\theta_1^Q & -\sin\theta_1^Q\\ \sin\theta_1^Q & \cos\theta_1^Q\end{bmatrix} \begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix} \begin{bmatrix}\cos\theta_1^Q & \sin\theta_1^Q\\ -\sin\theta_1^Q & \cos\theta_1^Q\end{bmatrix} \begin{bmatrix}\cos\alpha\\ \sin\alpha\end{bmatrix} =$$

$$= \begin{bmatrix}\cos(\alpha-\theta_1^Q)\\ \sin(\alpha-\theta_1^Q)\end{bmatrix}^T \begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix} \begin{bmatrix}\cos(\alpha-\theta_1^Q)\\ \sin(\alpha-\theta_1^Q)\end{bmatrix}. \tag{22}$$

Uvedemo li supstituciju

$$t = \alpha - \theta_1^Q \tag{23}$$

izraz (22) prelazi u

$$c^2(t) = \begin{bmatrix}\cos t\\ \sin t\end{bmatrix}^T \begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix} \begin{bmatrix}\cos t\\ \sin t\end{bmatrix} =$$

$$= \lambda_1 \cos^2 t + \lambda_2 \sin^2 t, \tag{24}$$

odnosno u

$$c^2(t) = \frac{1}{2}(e+g+K\cos 2t). \tag{25}$$

Budući da funkcija kosinus ima najveću vrijednost 1, a najmanju −1, to iz izraza (25) nije teško donijeti zaključak o najvećoj i najmanjoj vrijednosti linearnog mjerila $c = c(t)$. Najveće mjerilo označimo s A, a najmanje s B i imamo:

$$A^2 = \max c^2 = \lambda_1 = \frac{1}{2}(e+g+K) \text{ za } t = k\pi, k \in Z,$$

$$\text{odnosno } \alpha = \theta_1^Q + k\pi, k \in Z$$

$$B^2 = \min c^2 = \lambda_2 = \frac{1}{2}(e+g-K) \text{ za } t = \frac{\pi}{2} + k\pi, k \in Z,$$

$$\text{odnosno } \alpha = \theta_1^Q + \frac{\pi}{2} + k\pi, k \in Z. \tag{26}$$

Relacijom (21) određen je smjer najvećeg linearnog mjerila u promatranoj točki. Iz relacija (26) slijedi da smjer najmanjeg lineranog mjerila sa smjerom najvećega zatvara pravi kut. To je zapravo posljedica činjenice da su svojstveni vektori matrice međusobno okomiti. Smjerovi u kojima je linearno mjerilo najveće, odnosno najmanje, zovu se glavni smjerovi deformacija.

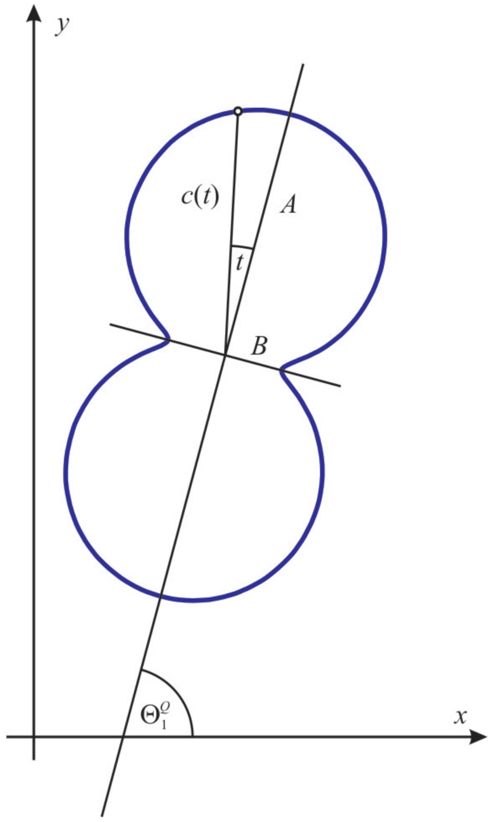

Figure 1. Curve of locale linear distortions

Slika 1. Krivulja lokalnih deformacija duljina

Ako se funkcija $c = c(\alpha)$ iz (11), odnosno $c = c(t)$ iz (24) ili (25) prikaže grafički (slika 1) dobije se krivulja koja se u teoriji pogrešaka naziva krivulja pogrešaka, podera ili nožišna krivulja elipse pogrešaka, u matematici podnožnica ili podera, a u teoriji kartografskih projekcija krivulja lokalnih deformacija duljina (grč. ποδηρης, franc. *podaire*, Lapaine 1989, 1991). Na temelju prethodnih izvoda zaključujemo da krivulju lokalnih deformacija duljina afinog preslikavanja možemo nacrtati u ravnini s pomoću jednadžbe u polarnom koordinatnom sustavu

$$c(t) = \sqrt{A^2\cos^2 t + B^2 \sin^2 t} \tag{27}$$

ili u pravokutnom koordinatnom sustavu x, y s pomoću parametarskih jednadžbi

$$\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}x(t)\\ y(t)\end{bmatrix} = c(t)\begin{bmatrix}\cos t\\ \sin t\end{bmatrix} = \sqrt{A^2\cos^2 t + B^2\sin^2 t}\begin{bmatrix}\cos t\\ \sin t\end{bmatrix} \tag{28}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + c(\alpha)\begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix} =$$

$$= \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \sqrt{e\cos^2\alpha + f\sin 2\alpha + g\sin^2\alpha}\begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}, \alpha \in [0, 2\pi].$$

2.2 Angles in affine mapping

Affine mapping generally does not preserve angles – except similarities. This can be proved as follows. The unit and orthogonal vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are mapped onto vectors $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ which, in general, are not unit vectors, and the angle θ between them is defined to the quadrant by the formula

$$\cos\theta = \frac{a_1 b_1 + a_2 b_2}{\sqrt{a_1^2 + a_2^2}\sqrt{b_1^2 + b_2^2}} = \frac{f}{\sqrt{eg}}. \tag{29}$$

In order for angle θ to be a right angle, it must be

$$f = a_1 b_1 + a_2 b_2 = 0 \tag{30}$$

which is not generally the case. Let us note that the condition (30) is necessary but not sufficient for an affine mapping preserves angles. Let us prove this.

If we use η to denote the angle between ds' in the plane of projection and the x' axis, then

$$\cos\eta = \frac{dx'}{ds'} = \frac{a_1 dx + b_1 dy}{\sqrt{edx^2 + 2fdxdy + gdy^2}},$$

$$\sin\eta = \pm\frac{a_2 dx + b_2 dy}{\sqrt{edx^2 + 2fdxdy + gdy^2}}, \tag{31}$$

$$\tan\eta = \pm\frac{a_2 dx + b_2 dy}{a_1 dx + b_1 dy}.$$

The sign in the formula (31) should be chosen in accordance with the quadrant of angle η. If $dy = 0$, then $\eta = \gamma$, giving

$$\cos\gamma = \frac{a_1}{\sqrt{e}}, \ \sin\gamma = \pm\frac{a_2}{\sqrt{e}}, \ \tan\gamma = \pm\frac{a_2}{a_1}. \tag{32}$$

If $dx = 0$, then $\eta = \gamma'$, giving

$$\cos\gamma' = \frac{b_1}{\sqrt{g}}, \ \sin\gamma' = \pm\frac{b_2}{\sqrt{g}}, \ \tan\gamma' = \pm\frac{b_2}{b_1}. \tag{33}$$

It should be remembered that the azimuth of the differential of arc in the plane xy was introduced by the relation (10). If we use β to denote the affine image of this azimuth in the $x'y'$ plane, then we can calculate that

$$\tan\beta = \tan(\eta - \gamma) = \frac{\tan\eta - \tan\gamma}{1 + \tan\eta\tan\gamma} = \tag{34}$$

$$= \frac{a_1(a_2 dx + b_2 dy) - a_2(a_1 dx + b_1 dy)}{a_1(a_1 dx + b_1 dy) + a_2(a_2 dx + b_2 dy)} = \frac{hdy}{edx + fdy} = \frac{h\tan\alpha}{e + f\tan\alpha}$$

from which it follows that

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} = \frac{(e-h)\tan\alpha + f\tan^2\alpha}{e + f\tan\alpha + h\tan^2\alpha}. \tag{35}$$

From the last relation (35) it follows that $\alpha = \beta$ if

$e = h$ and $f = 0$

which is equivalent to

$$e = g \text{ and } f = 0, \tag{36}$$

and this is the condition for conformal mapping, i.e. similarity. In other words, the regular affine mapping preserves angles if and only if (36) holds.

2.3 Linear scale of affine mapping in the image plane

Linear scale c can be expressed in the x', y' plane, too. In so doing, we need first of all to obtain from (2) the following:

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = M^{-1}\begin{bmatrix} dx' \\ dy' \end{bmatrix} = \frac{1}{a_1 b_2 - a_2 b_1}\begin{bmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{bmatrix}\begin{bmatrix} dx' \\ dy' \end{bmatrix}. \tag{37}$$

Let us denote it as is usual in differential geometry

$$h = \sqrt{eg - f^2} = |\det M| = |a_1 b_2 - a_2 b_1|, \tag{38}$$

This h can be explained as the area scale in affine mapping (see section 2.5). If we denote

$$P = \begin{bmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{bmatrix}^T \begin{bmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix}. \tag{39}$$

then

$$E = a_2^2 + b_2^2, \quad F = -(a_1 a_2 + b_1 b_2), \quad G = a_1^2 + b_1^2. \tag{40}$$

za $t \in [0, 2\pi]$, vodeći pritom računa o kutu θ_1^Q, jer je t definiran relacijom (23), a kut θ_1^Q se može izračunati primjenom formule (21). Budući da A i B ne ovise o promatranoj točki to zaključujemo da ni krivulja lokalnih deformacija ne ovisi o točki, u svakoj točki je jedna te ista.

Ako iz jednadžbi (28) eliminiramo parametar t možemo dobiti jednadžbu krivulje lokalnih deformacija u obliku

$$(x^2+y^2)^2 = A^2x^2 + B^2y^2$$

iz koje vidimo da je riječ o krivulji 4. reda. Ako želimo nacrtati krivulju lokalnih deformacija sa središem u točki s koordinatama x_0, y_0, a da pritom ne moramo voditi brigu o smjeru glavnih osi, tj. o kutu θ_1^Q, tada se možemo poslužiti ovim parametarskim jednadžbama

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + c(\alpha)\begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix} =$$

$$= \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \sqrt{e\cos^2\alpha + f\sin 2\alpha + g\sin^2\alpha}\begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}, \alpha \in [0, 2\pi].$$

2.2. Kutovi pri afinom preslikavanju

Afino preslikavanje općenito ne čuva kutove. U to se možemo uvjeriti ako uočimo da se jedinični vektori $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ preslikavaju u vektore $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ koji, naravno, općenito nisu jedinični, a kut θ među njima određen je do na kvadrant formulom

$$\cos\theta = \frac{a_1b_1 + a_2b_2}{\sqrt{a_1^2+a_2^2}\sqrt{b_1^2+b_2^2}} = \frac{f}{\sqrt{eg}}. \quad (29)$$

Da bi kut θ bio pravi, trebalo bi biti

$$f = a_1b_1 + a_2b_2 = 0 \quad (30)$$

što općenito nije slučaj. Nadalje, uvjet (30) je očito nužan da bi afino preslikavanje čuvalo kutove, ali nije i dovoljan. Dokažimo to.

Označimo s η kut između ds' u projekciji i osi x'. Tada je

$$\cos\eta = \frac{dx'}{ds'} = \frac{a_1dx + b_1dy}{\sqrt{edx^2 + 2fdxdy + gdy^2}},$$

$$\sin\eta = \pm\frac{a_2dx + b_2dy}{\sqrt{edx^2 + 2fdxdy + gdy^2}}, \quad (31)$$

$$\tan\eta = \pm\frac{a_2dx + b_2dy}{a_1dx + b_1dy}.$$

U formulama (31) predznak treba odabrati u skladu s kvadrantom u kojem je kut η. Ako je $dy = 0$, tada označimo $\eta = \gamma$ i imamo

$$\cos\gamma = \frac{a_1}{\sqrt{e}},\ \sin\gamma = \pm\frac{a_2}{\sqrt{e}},\ \tan\gamma = \pm\frac{a_2}{a_1}. \quad (32)$$

Ako je $dx = 0$, tada označimo $\eta = \gamma'$ i imamo

$$\cos\gamma' = \frac{b_1}{\sqrt{g}},\ \sin\gamma' = \pm\frac{b_2}{\sqrt{g}},\ \tan\gamma' = \pm\frac{b_2}{b_1}. \quad (33)$$

Prisjetimo se da smo s pomoću relacije (10) uveli azimut α proizvoljnog diferencijala luka krivulje u ravnini xy. Označimo s β afinu sliku toga azimuta u ravnini x', y'. Računamo

$$\tan\beta = \tan(\eta - \gamma) = \frac{\tan\eta - \tan\gamma}{1 + \tan\eta\tan\gamma} = \quad (34)$$

$$= \frac{a_1(a_2dx + b_2dy) - a_2(a_1dx + b_1dy)}{a_1(a_1dx + b_1dy) + a_2(a_2dx + b_2dy)} = \frac{hdy}{edx + fdy} = \frac{h\tan\alpha}{e + f\tan\alpha}$$

pa je

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} = \frac{(e-h)\tan\alpha + f\tan^2\alpha}{e + f\tan\alpha + h\tan^2\alpha}. \quad (35)$$

Iz posljednje relacije (35) slijedi da je $\alpha = \beta$ ako je

$$e = h \text{ i } f = 0$$

što je ekvivalentno s

$$e = g \text{ i } f = 0, \quad (36)$$

a to je uvjet za sličnost, odnosno za konformno preslikavanje. Drugim riječima, afino preslikavanje čuva kutove ako vrijedi (36).

2.3. Linearno mjerilo afinog preslikavanja u ravnini slike

Linearno mjerilo c može se izraziti i u ravnini x', y'. U tu svrhu iz (2) izrazimo najprije

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = M^{-1}\begin{bmatrix} dx' \\ dy' \end{bmatrix} = \frac{1}{a_1b_2 - a_2b_1}\begin{bmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{bmatrix}\begin{bmatrix} dx' \\ dy' \end{bmatrix}. \quad (37)$$

Označimo

$$h = \sqrt{eg - f^2} = |\det M| = |a_1b_2 - a_2b_1|, \quad (38)$$

Then we express

$$ds^2 = dx^2 + dy^2 = \begin{bmatrix} dx \\ dy \end{bmatrix}^T \begin{bmatrix} dx \\ dy \end{bmatrix} = \frac{1}{h^2} \begin{bmatrix} dx' \\ dy' \end{bmatrix}^T P \begin{bmatrix} dx' \\ dy' \end{bmatrix} =$$

$$= \frac{Edx'^2 + 2Fdx'dy' + Gdy'^2}{h^2}. \quad (41)$$

By substituting

$$\cos\eta = \frac{dx'}{ds'}, \sin\eta = \frac{dy'}{ds'} \quad (42)$$

(7) can be rewritten as

$$c^2(\eta) = \frac{ds'^2}{ds^2} = \frac{h^2}{\begin{bmatrix} \cos\eta \\ \sin\eta \end{bmatrix}^T P \begin{bmatrix} \cos\eta \\ \sin\eta \end{bmatrix}} = \frac{h^2}{E\cos^2\eta + F\sin 2\eta + G\sin^2\eta}. \quad (43)$$

Although the expression (43) is relatively simple, it is not easy or straightforward to derive a conclusion about the behaviour of the linear scale around the point in the image plane simply looking at the expression $c = c(\eta)$. So we will proceed as follows; firstly, instead of (43) we can write

$$h^2 c^{-2}(\eta) = \begin{bmatrix} \cos\eta \\ \sin\eta \end{bmatrix}^T P \begin{bmatrix} \cos\eta \\ \sin\eta \end{bmatrix} = E\cos^2\eta + F\sin 2\eta + G\sin^2\eta, \quad (44)$$

and then proceed analogously as in section 2.1. Any real symmetric matrix P can be factorized into a product

$$P = V \Lambda V^T, \quad (45)$$

where V is an orthogonal matrix, and Λ is a diagonal one. The columns of matrix V are orthonormal eigenvectors of matrix P, while the diagonal elements of matrix Λ are the corresponding eigenvalues. If we denote

$$V = \begin{bmatrix} \cos\theta_1^P & -\sin\theta_1^P \\ \sin\theta_1^P & \cos\theta_1^P \end{bmatrix} \text{ and } \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad (46)$$

then (45) is equivalent to

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} \lambda_1\cos^2\theta_1^P + \lambda_2\sin^2\theta_1^P & (\lambda_1 - \lambda_2)\sin\theta_1^P\cos\theta_1^P \\ (\lambda_1 - \lambda_2)\sin\theta_1^P\cos\theta_1^P & \lambda_1\sin^2\theta_1^P + \lambda_2\cos^2\theta_1^P \end{bmatrix}. \quad (47)$$

The expression (47) is equivalent to

$$\begin{aligned} &\lambda_1 \cos^2\theta_1^P + \lambda_2 \sin^2\theta_1^P = E \\ &\lambda_1 \sin^2\theta_1^P + \lambda_2 \cos^2\theta_1^P = G \\ &(\lambda_1 - \lambda_2)\sin\theta_1^P \cos\theta_1^P = F. \end{aligned} \quad (48)$$

From (48) it is easy to obtain

$$E + G = \lambda_1 + \lambda_2 \text{ i } EG - F^2 = \lambda_1 \lambda_2. \quad (49)$$

Since

$$E + G = e + g \text{ i } EG - F^2 = eg - f^2, \quad (50)$$

which is easy to check, it can be concluded that the eigenvalues of matrix P are equal to the eigenvalues of matrix Q, and this is why they are denoted by the same letters.

From the last equation in (48) it follows that

$$\sin 2\theta_1^P = \frac{2F}{K}, \cos 2\theta_1^P = \frac{E-G}{K}, \tan 2\theta_1^P = \frac{2F}{E-G} \quad (51)$$

and then

$$\tan\theta_1^P = \frac{\lambda_1^Q - E}{F} = \frac{F}{E - \lambda_2^Q} = \frac{1}{2F}(G - E + K). \quad (52)$$

Factorization (45) enables us to write (44) in the form

$$h^2 c^{-2}(\eta) = \begin{bmatrix} \cos\eta \\ \sin\eta \end{bmatrix}^T \begin{bmatrix} \cos\theta_1^P & -\sin\theta_1^P \\ \sin\theta_1^P & \cos\theta_1^P \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \cos\theta_1^P & \sin\theta_1^P \\ -\sin\theta_1^P & \cos\theta_1^P \end{bmatrix} \begin{bmatrix} \cos\eta \\ \sin\eta \end{bmatrix} =$$

$$= \begin{bmatrix} \cos(\eta - \theta_1^P) \\ \sin(\eta - \theta_1^P) \end{bmatrix}^T \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \cos(\eta - \theta_1^P) \\ \sin(\eta - \theta_1^P) \end{bmatrix}. \quad (53)$$

By substitution

$$\psi = \eta - \theta_1^P \quad (54)$$

the expression (53) proceeds to

$$h^2 c^{-2}(\psi) = \begin{bmatrix} \cos\psi \\ \sin\psi \end{bmatrix}^T \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \cos\psi \\ \sin\psi \end{bmatrix} = \lambda_1\cos^2\psi + \lambda_2\sin^2\psi, \quad (55)$$

and to

$$c^2(\psi) = \frac{2h^2}{e + g + K\cos 2\psi}. \quad (56)$$

Because the cosine function has a maximum value of 1, and a minimum value of –1, is not difficult to draw conclusions from the relation (48) on the maximum and minimum values of the linear scale $c = c(\psi)$, i.e. $c = c(\eta)$. If we denote the maximum scale with A, and minimum scale with B we get:

$$A^2 = \max c^2 = \lambda_1 = \frac{1}{2}(E + G + K) \text{ for } \psi = \frac{\pi}{2} + k\pi, k \in \mathbb{Z},$$

$$\text{i.e. } \eta = \theta_1^P + \frac{\pi}{2} + k\pi, k \in \mathbb{Z}$$

kao što je to uobičajeno u diferencijalnoj geometriji. Taj se h može protumačiti i kao mjerilo površina pri afinom preslikavanju (vidi poglavlje 2.5). Označimo

$$P=\begin{bmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{bmatrix}^T \begin{bmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{bmatrix}=\begin{bmatrix} E & F \\ F & G \end{bmatrix}. \tag{39}$$

Tada je

$$E=a_2^2+b_2^2,\quad F=-(a_1a_2+b_1b_2),\quad G=a_1^2+b_1^2. \tag{40}$$

Zatim izrazimo

$$ds^2=dx^2+dy^2=\begin{bmatrix} dx \\ dy \end{bmatrix}^T \begin{bmatrix} dx \\ dy \end{bmatrix}=\frac{1}{h^2}\begin{bmatrix} dx' \\ dy' \end{bmatrix}^T P \begin{bmatrix} dx' \\ dy' \end{bmatrix}=$$
$$=\frac{Edx'^2+2Fdx'dy'+Gdy'^2}{h^2}. \tag{41}$$

Ako uvedemo supstituciju

$$\cos\eta=\frac{dx'}{ds'},\ \sin\eta=\frac{dy'}{ds'} \tag{42}$$

možemo (7) dobiti u obliku

$$c^2(\eta)=\frac{ds'^2}{ds^2}=\frac{h^2}{\begin{bmatrix} \cos\eta \\ \sin\eta \end{bmatrix}^T P \begin{bmatrix} \cos\eta \\ \sin\eta \end{bmatrix}}=\frac{h^2}{E\cos^2\eta+F\sin 2\eta+G\sin^2\eta}. \tag{43}$$

Premda je izraz (43) relativno jednostavan, iz njega nije lako izravno izvesti zaključak o funkciji $c=c(\eta)$, tj. o ponašanju linearnog mjerila oko promatrane točke u ravnini slike. Stoga ćemo postupiti na sljedeći način. Najprije umjesto (43) napišimo

$$h^2c^{-2}(\eta)=\begin{bmatrix} \cos\eta \\ \sin\eta \end{bmatrix}^T P \begin{bmatrix} \cos\eta \\ \sin\eta \end{bmatrix}=E\cos^2\eta+F\sin 2\eta+G\sin^2\eta, \tag{44}$$

a zatim postupimo analogno kao u poglavlju 2.1. Svaka realna simetrična matrica P može se rastaviti na ovakav produkt

$$P=V\Lambda V^T, \tag{45}$$

gdje je V ortogonalna matrica, a Λ dijagonalna matrica. Pri tome su stupci matrice V ortonormirani svojstveni vektori matrice P, a dijagonalni elementi matrice Λ odgovarajuće svojstvene vrijednosti. Označimo li

$$V=\begin{bmatrix} \cos\theta_1^P & -\sin\theta_1^P \\ \sin\theta_1^P & \cos\theta_1^P \end{bmatrix} \text{ i } \Lambda=\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \tag{46}$$

onda je (45) ekvivalentno s

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix}=\begin{bmatrix} \lambda_1\cos^2\theta_1^P+\lambda_2\sin^2\theta_1^P & (\lambda_1-\lambda_2)\sin\theta_1^P\cos\theta_1^P \\ (\lambda_1-\lambda_2)\sin\theta_1^P\cos\theta_1^P & \lambda_1\sin^2\theta_1^P+\lambda_2\cos^2\theta_1^P \end{bmatrix}. \tag{47}$$

Izraz (47) je ekvivalentan s

$$\begin{aligned} &\lambda_1\cos^2\theta_1^P+\lambda_2\sin^2\theta_1^P=E \\ &\lambda_1\sin^2\theta_1^P+\lambda_2\cos^2\theta_1^P=G \\ &(\lambda_1-\lambda_2)\sin\theta_1^P\cos\theta_1^P=F. \end{aligned} \tag{48}$$

Iz (48) možemo lako dobiti

$$E+G=\lambda_1+\lambda_2 \ \text{ i } \ EG-F^2=\lambda_1\lambda_2. \tag{49}$$

Budući da vrijedi

$$E+G=e+g \ \text{ i } \ EG-F^2=eg-f^2, \tag{50}$$

što se lako može provjeriti, to zaključujemo da su svojstvene vrijednosti matrice P jednake svojstvenim vrijednostima matrice Q, pa smo ih stoga mogli označiti istim slovima.

Iz posljednje jednadžbe u (48) slijedi

$$\sin 2\theta_1^P=\frac{2F}{K},\ \cos 2\theta_1^P=\frac{E-G}{K},\ \tan 2\theta_1^P=\frac{2F}{E-G} \tag{51}$$

i zatim

$$\tan\theta_1^P=\frac{\lambda_1^Q-E}{F}=\frac{F}{E-\lambda_2^Q}=\frac{1}{2F}(G-E+K). \tag{52}$$

Rastav (45) omogućava da se (44) napiše u obliku

$$h^2c^{-2}(\eta)=\begin{bmatrix} \cos\eta \\ \sin\eta \end{bmatrix}^T \begin{bmatrix} \cos\theta_1^P & -\sin\theta_1^P \\ \sin\theta_1^P & \cos\theta_1^P \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \cos\theta_1^P & \sin\theta_1^P \\ -\sin\theta_1^P & \cos\theta_1^P \end{bmatrix} \begin{bmatrix} \cos\eta \\ \sin\eta \end{bmatrix}=$$
$$=\begin{bmatrix} \cos(\eta-\theta_1^P) \\ \sin(\eta-\theta_1^P) \end{bmatrix}^T \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \cos(\eta-\theta_1^P) \\ \sin(\eta-\theta_1^P) \end{bmatrix}. \tag{53}$$

Uvedemo li supstituciju

$$\psi=\eta-\theta_1^P \tag{54}$$

izraz (53) prelazi u

$$h^2c^{-2}(\psi)=\begin{bmatrix} \cos\psi \\ \sin\psi \end{bmatrix}^T \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \cos\psi \\ \sin\psi \end{bmatrix}=\lambda_1\cos^2\psi+\lambda_2\sin^2\psi, \tag{55}$$

odnosno u

$$c^2(\psi)=\frac{2h^2}{e+g+K\cos 2\psi}. \tag{56}$$

$$B^2 = \min c^2 = \lambda_2 = \frac{1}{2}(E+G-K) \text{ for } \psi = k\pi, k \in Z,$$
$$\text{i.e. } \eta = \theta_1^P + k\pi, k \in Z. \tag{57}$$

This justifies the introduction of the same letters, *A* and *B*, in (57) as in formulae (26). The direction of minimum linear scale at a given point in the image plane is defined by the relation (52). From (7), we see that the direction of maximum linear scale is orthogonal to the direction of minimum linear scale. Therefore, there are two main directions in the image plane which are mutually orthogonal.

If the function $c = c(\eta)$ from (43), or $c = c(\psi)$ from (56) is graphically represented (Figure 2) then we obtain an ellipse (Lapaine, 2006) which is known in the theory of map projections as the ellipse of distortion or Tissot's indicatrix (Lapaine 1991). On the base of the derivations given above, it can be concluded that the ellipse of linear distortions in the affine mapping can be visualized in the image plane by using the equation in the polar coordinate system

$$c^2(\psi) = \frac{A^2B^2}{A^2\cos^2\psi + B^2\sin^2\psi}, \tag{58}$$

or in the Cartesian coordinate system x', y' by using parametric equations

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x'(\psi) \\ y'(\psi) \end{bmatrix} = c(\psi)\begin{bmatrix} \cos\psi \\ \sin\psi \end{bmatrix} = \frac{AB}{\sqrt{A^2\cos^2\psi + B^2\sin^2\psi}}\begin{bmatrix} \cos\psi \\ \sin\psi \end{bmatrix} \tag{59}$$

with $\psi = [0, 2\pi]$, taking into account the angle θ_1^P, because the angle ψ was defined by (54), and the angle θ_1^P can be calculated by using (52). Since *A* and *B* do not depend on this point, it can be concluded that the ellipse of distortion does not depend on it either.

If we eliminate parameter ψ from (59) than we can get the equation

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$$

from which we see that it represents an ellipse with its centre in the origin, and having semiaxes *A* and *B*. If we want to visualize the ellipse of distortions with its centre at the point with coordinates x_0, y_0, and without taking into account the direction of main axes, then we can use the following parametric equations

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x'_0 \\ y'_0 \end{bmatrix} + \begin{bmatrix} x'(\eta) \\ y'(\eta) \end{bmatrix} = \begin{bmatrix} x'_0 \\ y'_0 \end{bmatrix} + c(\eta)\begin{bmatrix} \cos\eta \\ \sin\eta \end{bmatrix} =$$
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x'_0 \\ y'_0 \end{bmatrix} + \frac{h}{\sqrt{E\cos^2\eta + F\sin 2\eta + G\sin^2\eta}}\begin{bmatrix} \cos\eta \\ \sin\eta \end{bmatrix}, \eta \in [0, 2\pi].$$

2.4 Curve of local linear distortions and ellipse of distortions

The equation of the curve of locale linear distortions (27) is derived in section 2.1, and the equation of ellipse of distortion (59) is derived in section 3.3. Since both equations express the same phenomenon, local linear scale, it is only natural to raise the question of the relation between parameters *t* and ψ. The answer to this question can be obtained if we start from the relation

$$c(t) = c(\psi). \tag{60}$$

Taking into account (27) and (56), we get

$$A^2\cos^2 t + B^2\sin^2 t = \frac{A^2B^2}{A^2\sin^2\psi + B^2\cos^2\psi}, \tag{61}$$

and from there, the simple relation

$$|\tan t \tan\psi| = \frac{A}{B}. \tag{62}$$

For a given affine mapping, the values *A* and *B* are constants ((26) or (57)). Because the parameter *t* is defined by relation (18), and parameter ψ by (54), then (62) can be rewritten thus

$$\left|\tan(\alpha - \theta_1^Q)\tan(\eta - \theta_1^P)\right| = \frac{A}{B}. \tag{63}$$

It follows from (33) that for $\eta = \gamma$ it is $\alpha = 0$, and one of the consequences of the formula (63) is the relation

$$\left|\tan\theta_1^Q \tan(\theta_1^P - \gamma)\right| = \frac{A}{B}. \tag{64}$$

2.5 Scale of area in affine mapping

The area of a differential rectangle $A(x, y)$, $B(x+dx, y)$, $C(x+dx, y+dy)$, $D(x, y+dy)$ in the plane xOy is

$$dp = dxdy. \tag{65}$$

This rectangle can be mapped by the affine mapping given by (1) into the quadrangle with vertices

$$A'(a_1x + b_1y + c_1, a_2x + b_2y + c_2),$$
$$B'(a_1(x+dx) + b_1y + c_1, a_2(x+dx) + b_2y + c_2)),$$
$$C'(a_1(x+dx) + b_1(y+dy) + c_1, a_2(x+dx) + b_2(y+dy) + c_2),$$
$$D'(a_1x + b_1(y+dy) + c_1, a_2x + b_2(y+dy) + c_2). \tag{66}$$

Budući da funkcija kosinus ima najveću vrijednost 1, a najmanju −1, to iz izraza (48) nije teško donijeti zaključak o najvećoj i najmanjoj vrijednosti linearnog mjerila $c = c(\psi)$, odnosno $c = c(\eta)$. Najveće mjerilo označimo s A, a najmanje s B i imamo:

$$A^2 = \max c^2 = \lambda_1 = \frac{1}{2}(E+G+K) \ \text{za} \ \psi = \frac{\pi}{2} + k\pi, k \in Z,$$

$$\text{odnosno} \ \eta = \theta_1^P + \frac{\pi}{2} + k\pi, k \in Z$$

$$B^2 = \min c^2 = \lambda_2 = \frac{1}{2}(E+G-K) \ \text{za} \ \psi = k\pi, k \in Z,$$

$$\text{odnosno} \ \eta = \theta_1^P + k\pi, \ k \in Z. \tag{57}$$

To opravdava uvođenje oznaka A i B u (57) koje smo već imali u formulama (26). Relacijom (52) određen je smjer najmanjeg linearnog mjerila u promatranoj točki u ravnini slike. Smjer najvećeg lineranog mjerila sa smjerom najmanjega zatvara pravi kut (57). Dakle, u ravnini slike imamo također dva glavna smjera koji su međusobno okomiti.

Ako se funkcija $c = c(\eta)$ iz (43), odnosno $c = c(\psi)$ iz (56) prikaže grafički (slika 2) dobije se elipsa (Lapaine, 2006) koj$B$ se u teoriji kartografskih projekcija naziva elipsom deformacija ili Tissotovom indikatrisom (Lapaine 1991). Na temelju prethodnih izvoda zaključujemo da se elipsu deformacija duljina afinog preslikavanja može nacrtati u ravnini s pomoću jednadžbe u polarnom koordinatnom sustavu

$$c^2(\psi) = \frac{A^2 B^2}{A^2 \cos^2\psi + B^2 \sin^2\psi}, \tag{58}$$

ili u pravokutnom koordinatnom sustavu x', y' s pomoću parametarskih jednadžbi

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x'(\psi) \\ y'(\psi) \end{bmatrix} = c(\psi) \begin{bmatrix} \cos\psi \\ \sin\psi \end{bmatrix} = \frac{AB}{\sqrt{A^2\cos^2\psi + B^2\sin^2\psi}} \begin{bmatrix} \cos\psi \\ \sin\psi \end{bmatrix} \tag{59}$$

za $\psi \in [0, 2\pi]$, vodeći pritom računa o kutu θ_1^P, jer je ψ definiran relacijom (54), a kut θ_1^P se može izračunati primjenom formule (52). Budući da A i B ne ovise o promatranoj točki to zaključujemo da ni elipsa deformacija ne ovisi o točki, u svakoj točki je jedna te ista.

Ako iz jednadžbi (59) eliminiramo parametar ψ možemo dobiti jednadžbu u obliku

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$$

iz koje vidimo da je riječ o elipsi sa središtem u ishodištu i poluosima A i B. Ako želimo nacrtati elipsu deformacija sa središem u točki s koordinatama x_0, y_0, a da pritom ne moramo voditi brigu o smjeru glavnih osi, tj. o kutu θ_1^P, tada se možemo poslužiti ovim parametarskim jednadžbama

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x'_0 \\ y'_0 \end{bmatrix} + \begin{bmatrix} x'(\eta) \\ y'(\eta) \end{bmatrix} = \begin{bmatrix} x'_0 \\ y'_0 \end{bmatrix} + c(\eta) \begin{bmatrix} \cos\eta \\ \sin\eta \end{bmatrix} =$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x'_0 \\ y'_0 \end{bmatrix} + \frac{h}{\sqrt{E\cos^2\eta + F\sin 2\eta + G\sin^2\eta}} \begin{bmatrix} \cos\eta \\ \sin\eta \end{bmatrix}, \eta \in [0, 2\pi].$$

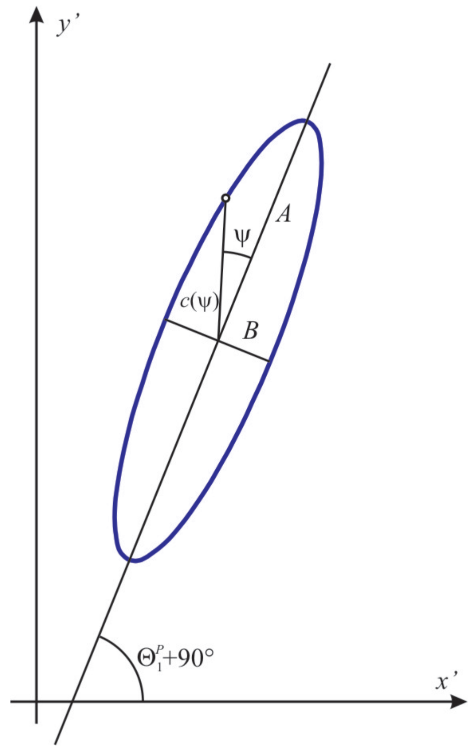

Figure 2. Ellipse of linear distortion in the image plane
Slika 2. Elipsa deformacija duljina u ravnini slike

2.4. Krivulja lokalnih deformacija duljina i elipsa deformacija

U poglavlju 2.1 izveli smo jednadžbu (27) krivulje lokalnih deformacija duljina, a u poglavlju 3.3 jednadžbu elipse deformacija (59). Budući da obje jednadžbe izražavaju isti fenomen, lokalno linearno mjerilo, prirodno se nameće pitanje koja je veza između parametara t i ψ. Odgovor na to pitanje dobit ćemo ako krenemo od izraza

$$c(t) = c(\psi). \tag{60}$$

Uzevši u obzir (27) i (56) dobit ćemo najprije

$$A^2 \cos^2 t + B^2 \sin^2 t = \frac{A^2 B^2}{A^2 \sin^2\psi + B^2 \cos^2\psi}, \tag{61}$$

i zatim odatle jednostavnu relaciju

$$|\tan t \tan\psi| = \frac{A}{B}. \tag{62}$$

The sides of that quadrangle are:

$$A'B' = C'D' = \sqrt{a_1^2 + a_2^2}\,dx = edx$$
$$B'C' = A'D' = \sqrt{b_1^2 + b_2^2}\,dy = gdy. \quad (67)$$

We can see that the opposite sides of the quadrangle $A'B'C'D'$ are equal and parallel. That means that the quadrangle $A'B'C'D'$ is a parallelogram. The angle θ between sides e and g is defined by the relation (29). The area of the parallelogram $A'B'C'D'$ is given by

$$dp' = edx\,gdy\sin\theta = eg\sqrt{1-\frac{f^2}{eg}}\,dxdy = \sqrt{eg-f^2}\,dxdy = hdxdy. \quad (68)$$

The scale of area p is generally defined in the following way

$$p = \frac{dp'}{dp} \quad (69)$$

which for the affine mapping of a plane onto a plane gives

$$p = h = AB. \quad (70)$$

We can see that the scale of area is independent of the point, but in contrast to the linear scale, it is also independent of direction. The scale of area is a constant of affine mapping and its value is defined by the formula (70). That means that any affine mapping is an equal-area mapping with the area scale h.

2.6 Example

If the equations of affine mapping of a plane onto a plane are given as

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 1 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

then according to (4), (19), (21) and (26), we can calculate that

$$e = 5, f = 10, g = 40, K = 5\sqrt{65},$$
$$A^2 = \frac{5(\sqrt{65}+9)}{2} = \frac{(\sqrt{65}+5)^2}{4}, A = \frac{\sqrt{65}+5}{2}$$
$$B^2 = \frac{5(9-\sqrt{65})}{2} = \frac{(\sqrt{65}-5)^2}{4}, B = \frac{\sqrt{65}-5}{2}$$
$$\frac{A}{B} = \frac{9+\sqrt{65}}{4} = 4{,}265$$
$$\tan\theta_1^Q = \frac{7+\sqrt{65}}{4} = 3{,}765.$$

The direction of the semiaxis A of the curve of local linear distortions regarding the coordinate axis x equals $\theta_1^Q = 75°08'$. Then, using the formula (31) we can obtain

$$\tan\gamma = \frac{1}{2} = 0{,}5\ ,\ \gamma = 26°34'.$$

This is the angle between the image of axis x and the coordinate axis x'.

According to (38) we can get h = 10, and by (52)

$$\tan\theta_1^P = \frac{29-5\sqrt{65}}{28} = -0{,}404\ ,\ \theta_1^P = -22°00'.$$

The direction of semiaxis A of the ellipse of distortion is .

$$\theta_1^P + 90° = 68°00'.$$

Now we are able to check the relation (64):

$$\tan(\theta_1^P - \gamma) = -\frac{1+\sqrt{65}}{8} = -1{,}133$$
$$\tan\theta_1^Q\tan(\theta_1^P - \gamma) = -\frac{9+\sqrt{65}}{4} = -4{,}265 = -\frac{A}{B}.$$

3 Ellipse of Distortion as the Image of a Unit Circle

One more geometric interpretation of the ellipse of distortion in the affine mapping of a plane into a plane will be given in this section. If a circle with its centre at the point (x_0, y_0) and radius 1 is given in the plane xOy, its equation reads

$$\begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix}^T \begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix} = 1. \quad (71)$$

We are looking for the image of this circle after affine mapping (1). If x, y and x', y' are the coordinates of a point and its image respectively, then it follows that

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = M\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \quad (72)$$

If x_0, y_0 and x_0', y_0' are the coordinates of a point and its image then it follows that

$$\begin{bmatrix} x_0' \\ y_0' \end{bmatrix} = M\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}. \quad (73)$$

If we subtract (73) from (72) we can get

$$\begin{bmatrix} x'-x_0' \\ y'-y_0' \end{bmatrix} = M\begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix} \quad (74)$$

and from there

Za zadano afino preslikavanje veličine A i B su konstante ((26), odnosno (57)). S obzirom na to da je parametar t definiran relacijom (18), a parametar ψ relacijom (54) to se (62) može napisati i ovako

$$\left|\tan(\alpha-\theta_1^Q)\tan(\eta-\theta_1^P)\right|=\frac{A}{B}. \tag{63}$$

Iz relacije (33) slijedi da je $\eta = \psi$ za $\alpha = 0$, pa je jedna posljedica formule (63) relacija

$$\left|\tan\theta_1^Q\tan(\theta_1^P-\gamma)\right|=\frac{A}{B}. \tag{64}$$

2.5. Mjerilo površina pri afinom preslikavanju

Površina diferencijalnog pravokutnika $A(x, y)$, $B(x+dx, y)$, $C(x+dx, y+dy)$, $D(x, y+dy)$ u ravnini xOy je

$$dp = dxdy\,. \tag{65}$$

Taj će se pravokutnik afinim preslikavanjem (1) preslikati u četverokut vrhovima

$$\begin{aligned}&A'(a_1x+b_1y+c_1,a_2x+b_2y+c_2),\\&B'(a_1(x+dx)+b_1y+c_1,a_2(x+dx)+b_2y+c_2)),\\&C'(a_1(x+dx)+b_1(y+dy)+c_1,a_2(x+dx)+b_2(y+dy)+c_2),\\&D'(a_1x+b_1(y+dy)+c_1,a_2x+b_2(y+dy)+c_2).\end{aligned} \tag{66}$$

Stranice toga četverokuta su:

$$\begin{aligned}A'B'=C'D'&=\sqrt{a_1^2+a_2^2}\,dx=edx\\B'C'=A'D'&=\sqrt{b_1^2+b_2^2}\,dy=gdy.\end{aligned} \tag{67}$$

Vidimo da su suprotne stranice četverokuta $A'B'C'D'$ međusobno jednake i paralelne. To znači da je četverokut $A'B'C'D'$ paralelogram. Kut θ između stranica e i g određen je relacijom (29). Površina paralelograma $A'B'C'D'$ je

$$dp'=edx\,gdy\sin\theta=eg\sqrt{1-\frac{f^2}{eg}}dxdy=\sqrt{eg-f^2}dxdy=hdxdy. \tag{68}$$

Mjerilo površina p definira se općenito ovako

$$p=\frac{dp'}{dp} \tag{69}$$

pa će za afino preslikavanje ravnine na ravninu biti

$$p=h=AB. \tag{70}$$

Dakle, mjerilo površina ne ovisi o promatranoj točki, ali za razliku od mjerila duljina, ni o smjeru. Ono je konstantno i po iznosu definirano formulom (70). Drugim riječima, svako afino preslikavanje je ekvivalentno u smislu teorije kartografskih projekcija, s mjerilom površina h.

2.6. Primjer

Neka su zadane jednadžbe afinog preslikavanja ravnine na ravninu u obliku

$$\begin{bmatrix}x'\\y'\end{bmatrix}=\begin{bmatrix}2&2\\1&6\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}1\\3\end{bmatrix}.$$

Prema formulama (4), (19), (21) i (26) možemo izračunati

$$e=5,f=10,g=40,K=5\sqrt{65},$$

$$A^2=\frac{5(\sqrt{65}+9)}{2}=\frac{(\sqrt{65}+5)^2}{4},A=\frac{\sqrt{65}+5}{2}$$

$$B^2=\frac{5(9-\sqrt{65})}{2}=\frac{(\sqrt{65}-5)^2}{4},B=\frac{\sqrt{65}-5}{2}$$

$$\frac{A}{B}=\frac{9+\sqrt{65}}{4}=4{,}265$$

$$\tan\theta_1^Q=\frac{7+\sqrt{65}}{4}=3{,}765.$$

Smjer poluosi A krivulje lokalnih deformacija duljina u odnosu na os x je $\theta_1^Q = 75°08'$. Zatim prema formuli (31) računamo

$$\tan\gamma=\frac{1}{2}=0{,}5\ ,\ \gamma=26°34'.$$

To je kut između slike osi x i osi x'. Prema (38) dobijemo $h = 10$, a prema (52)

$$\tan\theta_1^P=\frac{29-5\sqrt{65}}{28}=-0{,}404\ ,\ \theta_1^P=-22°00'.$$

Smjer polusi A elipse deformacija je dakle

$$\theta_1^P+90°=68°00'.$$

Sad je moguća provjera relacije (64):

$$\tan(\theta_1^P-\gamma)=-\frac{1+\sqrt{65}}{8}=-1{,}133$$

$$\tan\theta_1^Q\tan(\theta_1^P-\gamma)=-\frac{9+\sqrt{65}}{4}=-4{,}265=-\frac{A}{B}.$$

3. Elipsa deformacija kao slika jedinične kružnice

U ovom poglavlju dat ćemo još jednu geometrijsku interpretaciju elipse deformacija pri afinom preslikavanju ravnine na ravninu. Neka je u ravnini xOy zadana kružnica sa središtem u točki (x_0, y_0) i polumjerom 1. Njezina jednadžba glasi

$$\begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix} = M^{-1}\begin{bmatrix} x'-x_0' \\ y'-y_0' \end{bmatrix}. \tag{75}$$

Now, if we substitute (75) in (71), we get

$$\begin{bmatrix} x'-x_0' \\ y'-y_0' \end{bmatrix}^T P\begin{bmatrix} x'-x_0' \\ y'-y_0' \end{bmatrix} = h^2, \tag{76}$$

or

$$\begin{bmatrix} x'-x_0' \\ y'-y_0' \end{bmatrix}^T \begin{bmatrix} E & F \\ F & G \end{bmatrix}\begin{bmatrix} x'-x_0' \\ y'-y_0' \end{bmatrix} = h^2, \tag{77}$$

with the denotations introduced in formulae (39) and (40). By using

$$\begin{bmatrix} x'-x_0' \\ y'-y_0' \end{bmatrix} = c(\eta)\begin{bmatrix} \cos\eta \\ \sin\eta \end{bmatrix}, \tag{78}$$

(77) proceeds to (44), which we already know as an ellipse. The second possibility is to rewrite (77) as:

$$E(x'-x_0')^2 + 2F(x'-x_0')(y'-y_0') + G(y'-y_0')^2 = h^2 \tag{79}$$

and then conclude in the known way that this is the equation of a real ellipse with its centre at the point with coordinates (x_0', y_0'). According to Lapaine and Jovičić (1996), the semiaxes of the ellipse are given by

$$\sqrt{\frac{h^2}{B^2}} = A \quad \text{and} \quad \sqrt{\frac{h^2}{A^2}} = B. \tag{80}$$

The angle between the minor semiaxis of the ellipse and the coordinate axis x' is determined by the relation (52), while the major semiaxis is perpendicular to the minor one. Thus, we have proved that the ellipse of distortions can be interpreted as an image of a unit circle in affine mapping.

4 Conclusion

The affine transformation of coordinates from a plane onto a plane has applications in the different disciplines of surveying, geoinformatics and cartography, but generally speaking, there are only a few papers on the distortions that are unavoidable in such mappings. Hence, this paper discusses in detail distortions in affine mappings. The approach is taken from the theory of map projections. The appropriate formulae for the linear scale were derived, and on this basis it was seen that affine mapping generally does not preserve distances. Linear distortions depend on direction, and the distribution of linear distortions around a given point can be described by the curve of local linear distortions, or by the ellipse of distortions, depending on the point of view regarding the distribution of distortion in the plane to be mapped, or in its image. Next, ways of changing angles and areas in affine mapping were established. Finally, one possible interpretation of an ellipse of distortions as the image of a unit circle in affine mapping is given.

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$$\begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix}^T \begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix} = 1. \tag{71}$$

Potražimo sliku te kružnice pri afinom preslikavanju (1). Ako su x, y i x', y' koordinate točke i njezine slike onda vrijedi

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = M \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \tag{72}$$

Ako su x_0, y_0 i x_0', y_0' koordinate točke i njezine slike onda vrijedi

$$\begin{bmatrix} x_0' \\ y_0' \end{bmatrix} = M \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}. \tag{73}$$

Oduzmemo li (73) od (72) dobit ćemo

$$\begin{bmatrix} x'-x_0' \\ y'-y_0' \end{bmatrix} = M \begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix} \tag{74}$$

i odatle

$$\begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix} = M^{-1} \begin{bmatrix} x'-x_0' \\ y'-y_0' \end{bmatrix}. \tag{75}$$

Ako sad uvrstimo (75) u (71) dobit ćemo

$$\begin{bmatrix} x'-x_0' \\ y'-y_0' \end{bmatrix}^T P \begin{bmatrix} x'-x_0' \\ y'-y_0' \end{bmatrix} = h^2, \tag{76}$$

odnosno

$$\begin{bmatrix} x'-x_0' \\ y'-y_0' \end{bmatrix}^T \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} x'-x_0' \\ y'-y_0' \end{bmatrix} = h^2, \tag{77}$$

uz oznake uvedene u formulama (39) i (40). Uz

$$\begin{bmatrix} x'-x_0' \\ y'-y_0' \end{bmatrix} = c(\eta) \begin{bmatrix} \cos\eta \\ \sin\eta \end{bmatrix}, \tag{78}$$

(77) prelazi u (44), a to već znamo da je elipsa. Druga mogućnost je da (77) napišemo ovako:

$$E(x'-x_0')^2 + 2F(x'-x_0')(y'-y_0') + G(y'-y_0')^2 = h^2 \tag{79}$$

i onda na poznati način zaključimo da je riječ o (realnoj) elipsi sa središtem u točki s koordinatama (x_0', y_0'). Prema Lapaineu i Jovičiću (1996) poluosi te elipse su

$$\sqrt{\frac{h^2}{B^2}} = A \quad \text{i} \quad \sqrt{\frac{h^2}{A^2}} = B. \tag{80}$$

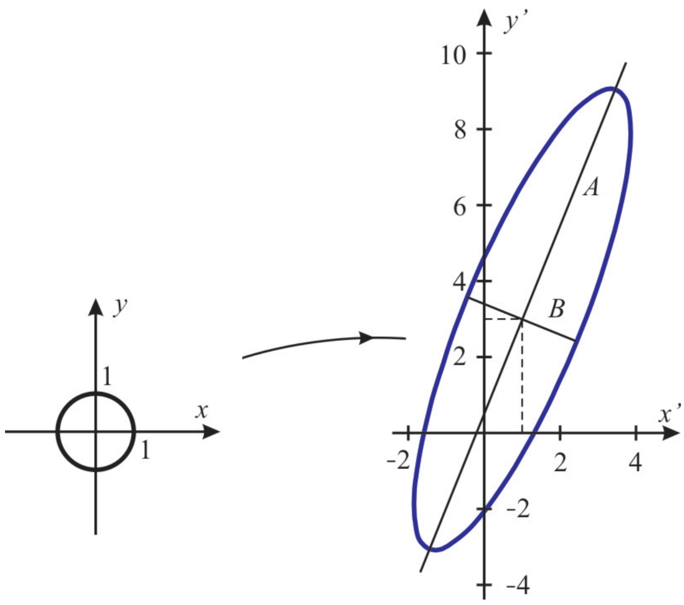

Figure 3. Ellipse of distortions as the image of a unit circle after affine mapping

Slika 3. Elipsa deformacija kao slika jedinične kružnice pri afinom preslikavanju

Kut koji manja poluos te elipse zatvara s koordinatnom osi x' određen je relacijom (52), a veća poluos je okomita na manju. Dakle, dokazano je da se elipsa deformacija može interpretirati i kao afina slika jedinične kružnice.

4. Zaključak

Afina transformacija koordinata ravnine na ravninu upotrebljava se u raznim područjima geodezije, geoinformatike i kartografije, ali se općenito malo govori o deformacijama koje neminovno sa sobom nosi svako takvo preslikavanje. Stoga je u ovome radu detaljno raspravljeno o deformacijama koje nastaju afinim preslikavanjem, a pristup toj problematici je analogan onome koji se primijenjuje u teoriji kartografskih projekcija. Izvedene su odgovarajuće formule za linearno mjerilo na temelju kojih je pokazano da se pri afinom preslikavanju općenito ne čuvaju udaljenosti. Deformacije duljina ovise o smjeru, a raspodjela deformacija duljina oko pojedine točke može se opisati s pomoću krivulje lokalnih deformacija duljina ili s pomoću elipse deformacija, ovisno o tome promatra li se ta raspodjela u ravnini koja se preslikava ili u ravnini na koju je preslikavanje obavljeno. Nadalje, utvrđeno je na koji se način mijenjaju kutovi i površine pri afinom preslikavanju ravnine na ravninu. Na kraju je prikazana jedna moguća interpretacija elipse deformacija kao afine slike jedinične kružnice.