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Loxodrome and Isometric Latitude

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Abstract. The paper gives a new generalized definition of the loxodrome on any surface. Its differential equations are derived, which are solved on the sphere and on the rotational ellipsoid. The concept of generalized longitude is introduced, which appears in a natural way when solving the differential equation of the loxodrome. Furthermore, the isometric latitude on the sphere and on the ellipsoid is introduced and show the relations with latitude and longitude. Generalized longitude allows defining isometric latitude in a new way. At the end, basic geodesic problems are solved along the loxodrome on the sphere and on the rotational ellipsoid.

Keywords: loxodrome, isometric latitude, generalized longitude, basic geodetic problems

1 Introduction

The term *loxodrome* comes from the Greek *loxos* (λοξός): slanting + *dromos* (δρόμος): road. In English, *loxodrome* and *rhumb line* are synonyms. The word *rhumb* probably comes from the Spanish/Portuguese *rumbo/rumo* (course, direction), Greek *rombos* (ρόμβος) or Latin *rhombus*: tern (Simović 1978, Lapaine 1993, Viher and Lapaine 1998, Lapaine 2006).

The first scientist who introduced the term loxodrome was Pedro Nunes (lat. *Petrus Nonius*, Alcácer do Sal, 1502 – Coimbra, 11 August 1578), a Portuguese mathematician, cosmographer and professor. He is considered one of the greatest mathematicians of his time. He is best known for his contributions to the navigation technique, which was extremely important at the time of the discovery of new territories. He is the inventor of several measuring devices, including the vernier (*nonius*), named after his Latin surname. Nunes was the first to understand why a ship that sails in a constant direction will not sail along the geodetic line, i.e. the shortest path connecting two points on Earth, but on a spiral course called a loxodrome. In the work *Tratado em defensam da carta de marear* (Treatise on the Defense of Maritime Charts) he advocated showing parallels and meridians as straight lines on navigational charts. However, he wasn't sure how to solve the problem. This question remained unsolved until Gerhard Mercator discovered a projection that is still used today and is called the Mercator projection after him (Simović 1978, Lapaine 1993, Viher and Lapaine 1998, Lapaine 2006).

Alexander (2004) mainly deals with the historical development and connection of the loxodrome with the Mercator projection. Petrović (2007) considers the loxodrome on the rotational ellipsoid, but only gives equations without a more detailed derivation and without concrete applications.

This article describes the results of the authors' research, which is based on Lapaine's 1993 article and is an

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Loksodroma i izometrijska širina

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S obzirom na to da su autori urednici ovog časopisa, postupak recenziranja obavio je i neovisnu uredničku odluku donio vanjski urednik prof. emer. Nedjeljko Frančula.
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Sažetak. U radu se daje nova poopćena definicija loksodrome na bilo kojoj plohi. Izvode se njezine diferencijalne jednaždbe koje se rješavaju na sferi i na rotacijskom elipsoidu. Uvodi se pojam poopćene geografske dužine, koja se pojavljuje na prirodan način pri rješavanju diferencijalne jednadžbe lokosodrome. Nadalje, uvodi se izometrijska širina na sferi i na elipsoidu, pokazuju relacije između nje i geografske širine te geografske dužine. Poopćena geografska dužina omogućuje definiranje izometrijske širine na nov način. Na kraju se rješavaju osnovni geodetski zadatci uzduž loksodrome na sferi i na rotacijskom elipsoidu.

Ključne riječi: loksodroma, izometrijska širina, poopćena geografska dužina, geodetski zadatci

1. Uvod

Riječ *loksodroma* dolazi od grčkoga *loxos* (λοξός): koso + *dromos* (δρόμος): put. Na engleskom jeziku susreću se kao sinonimi *loxodrome* i *rhumb line*. Riječ *rhumb* vjerojatno dolazi od španjolskoga/portugalskoga *rumbo/rumo* (kurs, smjer), grčkoga *rombos* (ρόμβος) ili latinskoga *rhombus*: čigra, zvrk (Simović 1978, Lapaine 1993, Viher i Lapaine 1998, Lapaine 2006).

Prvi znanstvenik koji je uveo pojam loksodrome bio je Pedro Nunes (lat. Petrus Nonius, Alcácer do Sal, 1502 – Coimbra, 11. 8. 1578), portugalski matematičar, kozmograf i profesor. Smatra se jednim od najvećih matematičara svojega doba. Najpoznatiji je po doprinosima u tehnici navigacije, što je bilo iznimno važno u to doba otkrića novih teritorija. Izumitelj je nekoliko mjernih sprava, među kojima je i *nonius*, nazvan prema njegovu latinskom prezimenu. Nunes je prvi shvatio zašto brod koji plovi stalnim smjerom neće ploviti uzduž geodetske linije, tj. najkraćeg puta koji spaja dvije točke na Zemlji, nego po spiralnom kursu koji se naziva loksodroma. U djelu *Tratado em defensam da carta de marear* (Rasprava o obrani pomorskih karata) zalagao se za prikazivanje paralela i meridijana u obliku ravnih crta na navigacijskim kartama. Međutim, nije bio siguran kako riješiti taj problem. To je pitanje ostalo neriješeno sve dok Gerhard Mercator nije otkrio projekciju koja se i danas upotrebljava i koja se po njemu naziva Mercatorovom projekcijom (Simović 1978, Lapaine 1993, Viher i Lapaine 1998, Lapaine 2006).

Alexander (2004) se pretežno bavi povijesnim razvojem i vezom loksodrome s Mercatorovom projekcijom. Petrović (2007) razmatra loksodromu na rotacijskom elipsoidu, ali daje samo jednadžbe bez detaljnijeg izvoda i bez konkretnih primjena.

Ovaj članak opisuje rezultate istraživanja autora koja se temelje na Lapaineovu članku iz 1993. godine i predstavljaju njegovu nadogradnju. Najprije se definira loksodroma na proizvoljnoj plohi, a zatim konkretizira extension of it. First, a loxodrome is defined on an arbitrary surface, and then it is specified on a sphere and on a rotational ellipsoid. It is shown that by analogy with the loxodrome on the sphere, the loxodrome on the ellipsoid can also be treated, only the formulas are more complex. In addition, the first and second geodesic problems for the loxodrome on the sphere and on the rotating ellipsoid are solved.

Let Ω be the area in the plane, and let $\vec{R}:\Omega\to R^3$ be the mapping of the area Ω into the three-dimensional space R^3. Let us mark:

$$\vec{R}(u,v) = (x,y,z),\quad x = x(u,v),\ y = y(u,v),\ z = z(u,v),\quad (u,v) \in \Omega,\ (x,y,z) \in R^3 \tag{1}$$

By mapping $\vec{R}$, a surface in space is defined. There, some other conditions are usually imposed on the mapping $\vec{R}$, for example, that the mapping is bijective, continuous, differentiable up to some order, etc. By definition, u-lines are curves for which v = const., while v-lines are curves for which u = const.

A straight line in a rectangular Cartesian system (u,v) has the well-known property that it closes equal angles with all u-lines (v-lines). The question naturally arises: is there such a curve on the surface that closes equal angles with all u-lines (v-lines)? If it exists, such a curve is called a *loxodrome.*

2 Definition and Equation of Loxodrome

Let $d\vec{R}$ be the differential of the mapping $\vec{R}$. It can be written $d\vec{R}=\vec{R}_u du+\vec{R}_v dv$. Then it is

$$d\vec{R}^2 = |d\vec{R}|^2 = ds^2 = \vec{R}_u^{\,2}du^2 + 2\vec{R}_u\vec{R}_v\,dudv + \vec{R}_v^{\,2}\,dv^2,$$

i.e. the first differential form of the mapping $\vec{R}$ is:

$$d\vec{R}^2 = |d\vec{R}|^2 = ds^2 = Edu^2 + 2Fdudv + Gdv^2,$$

where we marked as usual in mathematics:
$E = \vec{R}_u^{\,2}$, $F=\vec{R}_u\vec{R}_v$, $G=\vec{R}_v^{\,2}$. It is possible to interpret the first differential form as a cosine low for the triangle described by the relation $d\vec{R} = d\vec{R}_u + d\vec{R}_v = \vec{R}_u du + \vec{R}_v dv$.

Let $d\vec{R}$ denote the tangent vector to any curve, and $d\vec{R}_u$ the tangent vector to the u-curve at the observed point. Assume $d\vec{R}_u = \vec{R}_u du \neq \vec{0}$. Let us denote by α the angle between the curve and the u-curve and remember how this angle can be determined (Figure 1).

The scalar product of vectors $d\vec{R}$ and $d\vec{R}u$ is

$$d\vec{R}\cdot d\vec{R}_u = |d\vec{R}||d\vec{R}_u|\cos\alpha,$$

where:

$$|d\vec{R}| = ds,$$
$$|d\vec{R}_u| = \sqrt{d\vec{R}_u\, d\vec{R}_u} = \sqrt{\vec{R}_u\vec{R}_u}\,du = \sqrt{E}\,du,$$
$$(\vec{R}_u du + \vec{R}_v dv)\vec{R}_u du = \sqrt{E}\,\cos\alpha\, dsdu,$$
$$Edu^2 + Fdudv = \sqrt{E}\,\cos\alpha\, dsdu.$$

From there follows $Edu + Fdv = \sqrt{E}\cos\alpha ds$ because we assumed $d\vec{R}_u \neq \vec{0}$, and

$$\cos\alpha\, ds = \frac{Edu + Fdv}{\sqrt{E}}.$$

This expression is valid for any curve on the surface. Furthermore, according to the basic trigonometric relation $\sin^2\alpha + \cos^2\alpha = 1$ we have:

$$\sin^2\alpha = 1 - \cos^2\alpha = 1 - \left(\frac{Edu + Fdv}{\sqrt{E}}\right)^2 = \frac{EGdv^2 - F^2}{Eds^2},$$

and from there

$$\sin\alpha\, ds = \pm\frac{\sqrt{EG - F^2}}{\sqrt{E}}\,dv,$$

$$\tan\alpha == \pm\frac{\sqrt{EG - F^2}}{Edu + Fdv}\,dv.$$

Let us conclude that the following formulas are valid for any curve on the surface parametrized by the parameters u and v:

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2, \tag{2}$$

$$\cos\alpha\, ds = \frac{Edu + Fdv}{\sqrt{E}}, \tag{3}$$

$$\sin\alpha\, ds = \pm\frac{\sqrt{EG - F^2}}{\sqrt{E}}\,dv, \tag{4}$$

$$\tan\alpha == \pm\frac{\sqrt{EG - F^2}}{Edu + Fdv}\,dv. \tag{5}$$

Definition. Let α = const. If there is a solution of the differential equation (3), (4) or (5) in addition to (2), it is called a *loxodrome* on the surface defined by (1).

We leave the solving of this general task to those interested, and below we will solve simpler cases important for cartography and geodesy. Let us also note that equations (2)–(5) are not mutually independent. For example squaring and adding equations (3) and (4) yields (2), and dividing (4) and (3) yields (5).

na sferi i na rotacijskom elipsoidu. Pokazuje se da se po analogiji s loksodromom na sferi, može tretirati i loksodroma na elipsoidu, samo što su formule složenije. Osim toga, rješavaju se prvi i drugi geodetski zadatak za loksodromu na sferi i na rotacijskom elipsoidu.

Neka je Ω područje u ravnini, a $\vec{R}:\Omega\to R^3$ preslikavanje područja Ω u trodimenzionalan prostor R^3. Označimo:

$$\begin{aligned}&\vec{R}(u,v) = (x,y,z),\\ &x = x(u,v),\ y = y(u,v),\ z = z(u,v) \qquad (1)\\ &(u,v) \in \Omega,\ (x,y,z) \in R^3\end{aligned}$$

Preslikavanjem $\vec{R}$ definirana je ploha u prostoru. Tu se obično zadaju još neki uvjeti na preslikavanje $\vec{R}$, npr. da je to preslikavanje bijektivno, neprekidno, diferencijabilno do nekog reda itd. Po definiciji u-linije su takve krivulje za koje je v=const., dok su v-linije takve krivulje za koje je u=const.

Pravac u ravnini u pravokutnom kartezijevom sustavu (u,v) ima poznato svojstvo da sa svim u-linijama (v-linijama) zatvara jednake kutove. Prirodno se postavlja pitanje: postoji li na plohi takva krivulja koja sa svim u-linijama (v-linijama) zatvara jednake kutove? Ako postoji, takva se krivulja naziva *loksodromom*.

2. Definicija i jednadžba loksodrome

Neka je $d\vec{R}$ diferencijal preslikavanja $\vec{R}$. Može se napisati $d\vec{R} = \vec{R}_u\,du + \vec{R}_v\,dv$. Tada je

$$d\vec{R}^2 = |d\vec{R}|^2 = ds^2 = \vec{R}_u^{\,2}\,du^2 + 2\vec{R}_u\vec{R}_v\,dudv + \vec{R}_v^{\,2}\,dv^2,$$

tj. prva diferencijalna forma preslikavanja $\vec{R}$ je:

$$d\vec{R}^2 = |d\mathrm{R}|^2 = ds^2 = Edu^2 + 2Fdudv + Gdv^2,$$

gdje smo označili kao što je uobičajeno u matematici: $E = \vec{R}_u^{\,2}$, $F = \vec{R}_u\vec{R}_v$, $G=\vec{R}_v^{\,2}$. Moguća je interpretacija prve diferencijalne forme kao kosinusnog poučka za trokut koji opisuje relacija $d\vec{R} = d\vec{R}_u + d\vec{R}_v = \vec{R}_u\,du + \vec{R}_v\,dv$.

Označimo s $d\vec{R}$ vektor tangente na bilo koju krivulju, a $d\vec{R}_u$ vektor tangente na u-krivulju u promatranoj točki. Pretpostavimo $d\vec{R}_u = \vec{R}_u\,du \neq \vec{0}$. Označimo s α kut između krivulje i u-krivulje i podsjetimo se kako se taj kut može odrediti (slika 1).

Skalarni produkt vektora $d\vec{R}$ i $d\vec{R}_u$ je

$$d\vec{R}\cdot d\vec{R}_u = |d\vec{R}||d\vec{R}_u|\cos\alpha,$$

gdje su:

$$|d\vec{R}| = ds,$$

Slika 1. Vektori $d\vec{R}, d\vec{R}_u$ u nekoj točki na plohi i kut α kut između neke krivulje i u-krivulje.
Fig. 1 Vectors $d\vec{R}, d\vec{R}_u$ at some point on the surface and angle α is the angle between some curve and u-curve.

$$\begin{aligned}&|d\vec{R}_u| = \sqrt{d\vec{R}_u\,d\vec{R}_u} = \sqrt{\vec{R}_u\vec{R}_u}\,du = \sqrt{E}\,du,\\ &(\vec{R}_u du + \vec{R}_v dv)\vec{R}_u du = \sqrt{E}\ \cos\alpha\, ds du,\\ &Edu^2 + Fdudv = \sqrt{E}\ \cos\alpha\, ds du.\end{aligned}$$

Odatle slijedi $Edu + Fdv = \sqrt{E}\cos\alpha ds$ jer smo pretpostavili $d\vec{R}_u \neq \vec{0}$, i zatim

$$\cos\alpha\ ds = \frac{Edu + Fdv}{\sqrt{E}}\ .$$

Taj izraz vrijedi za bilo koju krivulju na plohi. Nadalje, prema osnovnoj trigonometrijskoj relaciji $\sin^2\alpha + \cos^2\alpha = 1$ imamo:

$$\sin^2\alpha = 1 - \cos^2\alpha = 1 - \left(\frac{Edu + Fdv}{\sqrt{E}}\right)^2 = \frac{EGdv^2 - F^2}{Eds^2},$$

i odatle

$$\sin\alpha\ ds = \pm\frac{\sqrt{EG - F^2}}{\sqrt{E}}\,dv,$$

$$\tan\alpha == \pm\frac{\sqrt{EG - F^2}}{Edu + Fdv}\,dv.$$

Zaključimo da sljedeće formule vrijede za bilo koju krivulju na plohi parametriziranoj parametrima u i v:

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2, \qquad (2)$$

$$\cos\alpha\ ds = \frac{Edu + Fdv}{\sqrt{E}}\ , \qquad (3)$$

$$\sin\alpha\ ds = \pm\frac{\sqrt{EG - F^2}}{\sqrt{E}}\,dv, \qquad (4)$$

$$\tan\alpha == \pm\frac{\sqrt{EG - F^2}}{Edu + Fdv}\,dv. \qquad (5)$$

3 The equation of the curve on the surface where *u*-lines and *v*-lines form an orthogonal network

Let us take the special case of a surface for which $F = \vec{R}_u \vec{R}_v = 0$, i.e. let the *u*-curves and *v*-curves intersect at right angles. Expressions (2)–(5), which are valid for any curve on the surface, for the surface for which $F = 0$ read like this (right triangle, theorem of Pythagora):

$$ds^2 = Edu^2 + Gdv^2, \tag{6}$$
$$\cos\alpha\, ds = \sqrt{E}\, du, \tag{7}$$
$$\sin\alpha\, ds = \pm\sqrt{G}\, dv, \tag{8}$$
$$\tan\alpha\, ds == \pm\sqrt{\frac{G}{E}}\frac{dv}{du}. \tag{9}$$

If still α = const. then (7), (8) and (9) along with (6) are the differential equations of the loxodrome in three different forms. We will consider the special cases $\alpha = k\pi$, $k \in Z$ and $\alpha = k\frac{\pi}{2}$, $k \in Z$ in section 7.

4 The Equation of the Loxodrome on the Sphere

Since in geodesy and cartography parameters are usually not denoted by *u* and *v*, but by φ and λ, instead of *u* and *v* in the following part of the paper we will use φ and λ. Let us recall that for R = const. the mapping

$$\vec{R}(\varphi,\lambda) = (x, y, z),$$
$$x = x(\varphi,\lambda) = R\cos\varphi\cos\lambda,$$
$$y = y(\varphi,\lambda) = R\cos\varphi\sin\lambda,$$
$$z = z(\varphi,\lambda) = R\sin\varphi,$$
$$(\varphi,\lambda) \in \Omega = [-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\pi,\pi],\ (x,y,z) \in \mathrm{R}^3$$

defines a sphere with its center at the origin of the coordinate system and radius R.

The coefficients of the first differential form of this mapping are:

$$E = R^2,\ F = 0,\ G = R^2\cos^2\varphi.$$

Accordingly, the differential expressions (6)–(9) for any curve on the sphere are:

$$ds^2 = R^2 d\varphi^2 + R^2\cos^2\varphi\, d\lambda^2, \tag{10}$$
$$\cos\alpha\, ds = R d\varphi, \tag{11}$$
$$\sin\alpha\, ds = \pm R\cos\varphi\, d\lambda, \tag{12}$$
$$\tan\alpha = \pm\cos\varphi\frac{d\lambda}{d\varphi}. \tag{13}$$

Let us agree that the angle α will have a value from the interval $(0,2\pi)$. It is the azimuth that will be measured clockwise so that the relationships shown in Table 1 apply.

Tablica 1. Osnovni odnosi između geografske širine, geografske dužine i azimuta.

Table 1 Basic relationships between latitude, longitude and azimuth.

$\varphi_1 < \varphi_2$	$\lambda_1 < \lambda_2$	$\alpha \in \left(0, \frac{\pi}{2}\right)$
$\varphi_1 > \varphi_2$	$\lambda_1 < \lambda_2$	$\alpha \in \left(\frac{\pi}{2}, \pi\right)$
$\varphi_1 > \varphi_2$	$\lambda_1 > \lambda_2$	$\alpha \in \left(\pi, \frac{3\pi}{2}\right)$
$\varphi_1 < \varphi_2$	$\lambda_1 > \lambda_2$	$\alpha \in \left(\frac{3\pi}{2}, 2\pi\right)$

If we accept the relations from Table 1, and the fact that the length of the arc from the curve must be positive, then in formulas (12) and (13) it is sufficient to take only the positive sign.

Let α = const. The differential equation of the loxodrome on the sphere is then e.g. (11), and can be solved in the following simple way:

$$\cos\alpha \int ds = R\int d\varphi,$$

which after integration gives

$$s\cos\alpha = R(\varphi - \varphi_1), \tag{14}$$

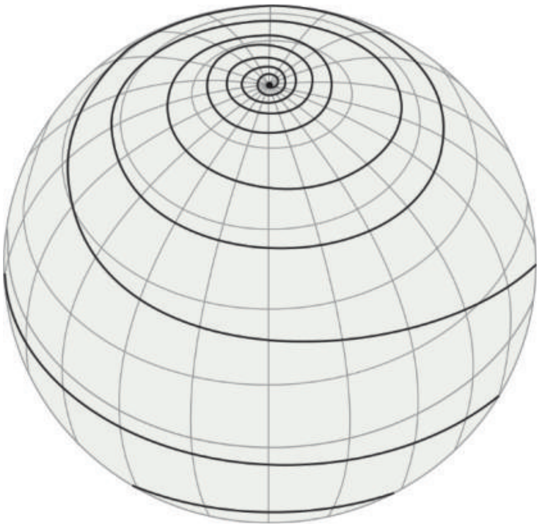
namely the loxodrome equation that connects the latitude φ and the arc length s. That loxodrome passes through a point with latitude φ_1 and at that point the arc length is 0.

Loxodromes on a sphere are generally spiral curves that wrap around each pole an infinite number of times (Figures 4 and 5), and as we shall soon see, never reach it, although their length is finite.

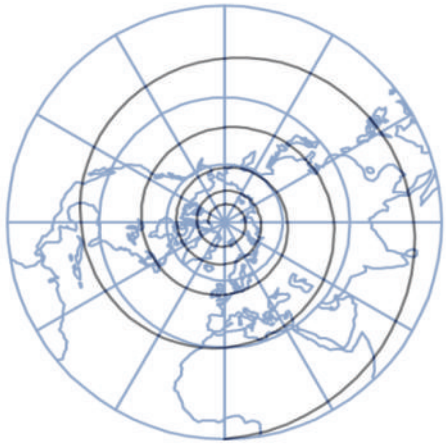
The length of the loxodrome from pole to pole is equal to the length of the arc of the meridian divided by the cosine of the angle α. Indeed, in formula (14) we should put $\varphi_1 = -\pi/2$, $s_1 = 0$, $\varphi = \pi/2$, so we obtain $s = R\pi/\cos\alpha$, $\alpha \neq \pi/2 + k\pi$, $k \in Z$.

If we start with the differential equation (12), we cannot integrate it immediately, but first we should express φ s by means of λ or s. Therefore, we prefer to take equation (13), which can be integrated if we write it in the form

$$d\lambda = \tan\alpha\frac{d\varphi}{\cos\varphi}. \tag{15}$$



Slika 2. Loksodroma na sferi.
Fig. 2 Loxodrome on a sphere.



Slika 3. Loksodroma u stereografskoj projekciji sjeverne polusfere.
Fig. 3 Loxodrome in the stereographic projection of the northern hemisphere.

Definicija. Neka je α = const. Ako postoji rješenje diferencijalne jednadžbe (3), (4) ili (5) uz (2), ono se zove *loksodroma* na plohi definiranoj s (1).

Rješavanje tog općenitog zadatka prepuštamo zainteresiranima, a mi ćemo u nastavku riješiti jednostavnije slučajeve važne za kartografiju i geodeziju. Uočimo još da jednadžbe (2)–(5) nisu međusobno nezavisne. Npr. kvadriranjem i zbrajanjem jednadžbi (3) i (4) dobije se (2), a dijeljenjem (4) i (3) dobije se (5).

3. Jednadžba krivulje na plohi na kojoj *u*-linije i *v*-linije čine ortogonalnu mrežu

Uzmimo poseban slučaj plohe za koju vrijedi $F = \vec{R}_u \vec{R}_v = 0$, tj. neka se u-krivulje i v-krivulje sijeku pod pravim kutom. Izrazi (2)–(5), koji vrijede za bilo koju krivulju na plohi, za plohu za koju je $F = 0$ glase ovako (pravokutan trokut, Pitagorin poučak):

$$ds^2 = Edu^2 + Gdv^2, \tag{6}$$

$$\cos\alpha\, ds = \sqrt{E}\, du, \tag{7}$$

$$\sin\alpha\, ds = \pm \sqrt{G}\, dv, \tag{8}$$

$$\tan\alpha\, ds == \pm \sqrt{\frac{G}{E}} \frac{dv}{du}. \tag{9}$$

Ako je još $\alpha = const.$ tada su (7), (8) i (9) uz (6) diferencijalne jednadžbe loksodrome u tri različita oblika. Posebne slučajeve $\alpha = k\pi$, $k \in Z$ i $\alpha = k\frac{\pi}{2}$, $k \in Z$ razmatrat ćemo u poglavlju 7.

4. Jednadžba loksodrome na sferi

Budući da se u geodeziji i kartografiji parametri obično ne označavaju s u i v, nego s φ i λ, umjesto u i v u tekstu koji slijedi upotrebljavat ćemo φ i λ. Prisjetimo se da se za R = const. preslikavanjem

$$\begin{aligned}
&\vec{R}(\varphi, \lambda) = (x, y, z),\\
&x = x(\varphi, \lambda) = R\cos\varphi\cos\lambda,\\
&y = y(\varphi, \lambda) = R\cos\varphi\sin\lambda,\\
&z = z(\varphi, \lambda) = R\sin\varphi,\\
&(\varphi, \lambda) \in \Omega = [-\tfrac{\pi}{2}, \tfrac{\pi}{2}] \times [-\pi, \pi], (x,y,z) \in \mathrm{R}^3
\end{aligned}$$

definira sfera sa središtem u ishodištu koordinatnog sustava i polumjerom R.

Koeficijenti prve diferencijalne forme toga preslikavanja su:

$$E = R^2, F = 0, G = R^2 \cos^2\varphi.$$

Prema tome diferencijalni izrazi (6)–(9) za bilo koju krivulju na sferi su:

$$ds^2 = R^2 d\varphi^2 + R^2 \cos^2\varphi\, d\lambda^2, \tag{10}$$

$$\cos\alpha\, ds = R d\varphi, \tag{11}$$

$$\sin\alpha\, ds = \pm R\cos\varphi\, d\lambda, \tag{12}$$

$$\tan\alpha = \pm \cos\varphi \frac{d\lambda}{d\varphi}. \tag{13}$$

Dogovorimo se da će kut α imati vrijednost iz intervala $(0,2\pi)$. To je azimut koji će se mjeriti u

After integration we get

$$\lambda = \tan\alpha \, \ln\tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right) + \beta = \tan\alpha \, \tanh^{-1}(\sin\varphi) + \beta. \quad (16)$$

If we want the loxodrome to pass through the point with geographical coordinates (φ_1, λ_1), it is necessary to take the integration constant β

$$\beta = \lambda_1 - \tan\alpha \, \ln\tan\left(\frac{\pi}{4}+\frac{\varphi_1}{2}\right) = \lambda_1 - \tan\alpha \, \tanh^{-1}(\sin\varphi_1). \quad (17)$$

Finally, if we want the relationship between λ and s we can write

$$\lambda = \tan\alpha \, \ln\tan\left(\frac{\pi}{4}+\frac{s\cos\alpha}{2R}+\frac{\varphi_1}{2}\right) + \beta = \tan\alpha \, \tanh^{-1}\left(\sin\left(\frac{s\cos\alpha}{R}+\varphi_1\right)\right) + \beta.$$

If $s = 0$, then $\lambda = \lambda_1$.

Figure 3 shows part of the loxodrome in the northern hemisphere in the stereographic projection. The equation of that loxodrome in the polar coordinate system in the plane of the stereographic projection is:

$$\rho = K\tan\left(\frac{\pi}{4}-\frac{\varphi}{2}\right) = \frac{K}{\tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)} = K\exp\left(-\frac{\lambda}{\tan\alpha}\right), \alpha \neq 0.$$

In mathematics, such a curve is called a logarithmic spiral.

5 Isometric Latitude and Loxodrome

The isometric latitude q on the sphere in the theory of map projections is defined by the geographic latitude φ and the differential equation

$$dq = \frac{d\varphi}{\cos\varphi}. \quad (18)$$

The solution of differential equation (18) is

$$q = \ln\tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right) = \frac{1}{2}\ln\left(\frac{1+\sin\varphi}{1-\sin\varphi}\right) = \tanh^{-1}(\sin\varphi). \quad (19)$$

with the assumption that for the integration constant we took the value that gives $\varphi = 0$ for $q = 0$. The inverse equation is:

$$\varphi = \sin^{-1}(\tanh q) = 2\tan^{-1}(\exp(q)) - \frac{\pi}{2}. \quad (20)$$

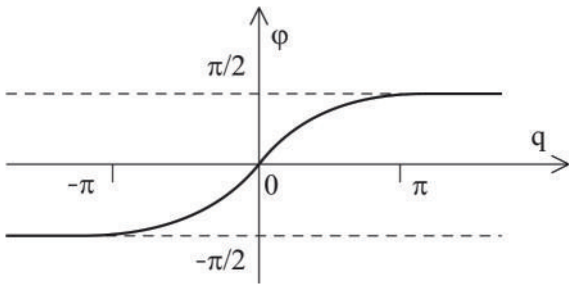


Fig. 4 Relationship between geographic and isometric latitude. The geographic latitude $\varphi\in(-\pi/2,\pi/2)$ corresponds to the isometric latitude $q\in(-\infty,\infty)$ and vice versa, formulas (19) and (20).

Slika 4. Odnos između geografske i izometrijske širine. Geografskoj širini $\varphi\in(-\pi/2,\pi/2)$, odgovara izometrijska širina $q\in(-\infty,\infty)$ i obratno, formule (19) i (20).

A graphic representation of the mutual dependence of geographic and isometric latitude is given in Figure 4.

The following relations are easily derived from the definition of isometric latitude

$$\tanh q = \sin\varphi, \ \sinh q = \tan\varphi, \ \cosh q = \frac{1}{\cos\varphi}, \quad \tanh\frac{q}{2} = \tan\frac{\varphi}{2}, \ \exp(q) = \tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right). \quad (21)$$

Furthermore, the differential equation (15) written by using the isometric latitude q becomes very simple and reads

$$d\lambda = \tan\alpha \, dq.$$

After integration, we get the equation of the loxodrome on the sphere in the form

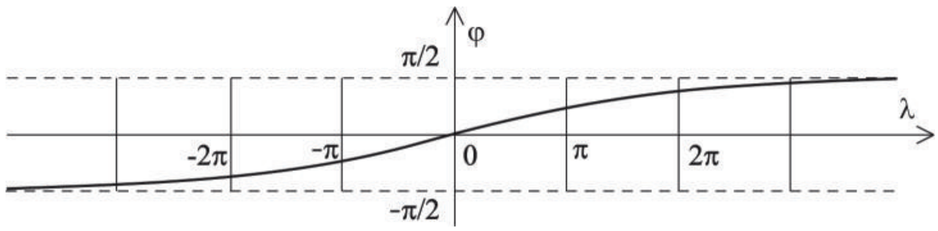
$$\lambda = q\tan\alpha + \beta, \quad (23)$$

where β is the integration constant. If we want the loxodrome to pass through the point with coordinates (q_1, λ_1), it is necessary to take the integration constant β

$$\beta = \lambda_1 - q_1\tan\alpha. \quad (24)$$

6 Generalized Longitude λ

We notice that for any $\varphi\in(-\pi/2,\pi/2)$, λ determined according to (16) is a real number that generally does not have to be from the interval $(-\pi,\pi)$ and does not



Slika 5. Loksodroma u koordinatnom sustavu λ, φ, gdje je λ geografska dužina u širem smislu, a $\beta = 0$.
Fig. 5 Loxodrome in the coordinate system λ, φ, where λ is longitude in the broad sense, and $\beta = 0$.

smjeru kazaljke na satu tako da vrijede odnosi prikazani u tablici 1.

Prihvatimo li odnose iz tablice 1, te činjenicu da duljina luka s krivulje mora biti pozitivna, tada je u formulama (12) i (13) dovoljno uzeti samo pozitivan predznak.

Neka je α = *const*. Diferencijalna jednadžba loksodrome na sferi je tada npr. (11), a može se riješiti na sljedeći jednostavan način:

$$\cos\alpha \int ds = R\int d\varphi,$$

što nakon integracije daje

$$s\cos\alpha = R(\varphi - \varphi_1), \tag{14}$$

i to jednadžba loksodrome koja povezuje geografsku širinu φ i duljinu luka s. Ta loksodroma prolazi točkom s geografskom širinom φ_1 i u toj je točki duljina luka 0.

Loksodrome na sferi su općenito spiralne krivulje koje se omataju oko svakog pola beskonačno mnogo puta (slike 4 i 5), i kao što ćemo uskoro vidjeti, do njega nikad ne stižu, premda je njihova duljina konačna. Duljina loksodrome od pola do pola jednaka je duljini luka meridijana podijeljenog kosinusom kuta α. Zaista, u formulu (14) treba staviti $\varphi_1 = -\pi/2$, $s_1 = 0$, $\varphi = \pi/2$, pa se dobije $s = R\pi/\cos\alpha$, $\alpha \neq \pi/2 + k\pi$, $k \in Z$.

Ako krenemo s diferencijalnom jednadžbom (12) ne možemo je odmah integrirati, nego bi najprije trebalo izraziti φ s pomoću λ ili s. Stoga ćemo radije uzeti jednadžbu (13) koja se može integrirati ako je napišemo u obliku

$$d\lambda = \tan\alpha \frac{d\varphi}{\cos\varphi}. \tag{15}$$

Nakon integriranja dobijemo

$$\lambda = \tan\alpha \ \ln\tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right) + \beta = \tan\alpha \ \tanh^{-1}(\sin\varphi) + \beta. \tag{16}$$

Ako želimo da loksodroma prolazi točkom s geografskim koordinatama (φ_1, λ_1), potrebno je za integracijsku konstantu β uzeti

$$\beta = \lambda_1 - \tan\alpha \ \ln\tan\left(\frac{\pi}{4} + \frac{\varphi_1}{2}\right) = \lambda_1 - \tan\alpha \ \tanh^{-1}(\sin\varphi_1). \tag{17}$$

Konačno, ako želimo vezu između λ i s možemo napisati

$$\lambda = \tan\alpha \ \ln\tan\left(\frac{\pi}{4} + \frac{s\cos\alpha}{2R} + \frac{\varphi_1}{2}\right) + \beta =$$
$$\tan\alpha \ \tanh^{-1}\left(\sin\left(\frac{s\cos\alpha}{R} + \varphi_1\right)\right) + \beta.$$

Ako je $s = 0$, onda je $\lambda = \lambda_1$.

Na slici 3 prikazan je dio loksodrome na sjevernoj polusferi u stereografskoj projekciji. Jednadžba te loksodrome u polarnom koordinatnom sustavu u ravnini stereografske projekcije glasi

$$\rho = K\tan\left(\frac{\pi}{4} - \frac{\varphi}{2}\right) = \frac{K}{\tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)} = K\exp\left(-\frac{\lambda}{\tan\alpha}\right), \ \alpha \neq 0.$$

U matematici se takva krivulja naziva logaritamskom spiralom.

5. Izometrijska širina i loksodroma

Izometrijsku širinu q na sferi u teoriji kartografskih projekcija definiramo s pomoću geografske širine φ i diferencijalne jednadžbe

$$dq = \frac{d\varphi}{\cos\varphi}. \tag{18}$$

Rješenje diferencijalne jednaždbe (18) je

$$q = \ln\tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right) = \frac{1}{2} \ln\left(\frac{1 + \sin\varphi}{1 - \sin\varphi}\right) = \tanh^{-1}(\sin\varphi). \tag{19}$$

uz pretpostavku da smo za integracijsku konstantu uzeli onu vrijednost koja za $\varphi = 0$ daje $q = 0$. Obratna veza je:

$$\varphi = \sin^{-1}(\tanh q) = 2\tan^{-1}(\exp(q)) - \frac{\pi}{2}. \tag{20}$$

have the meaning of longitude in the usual sense (Figure 5). This is also clearly seen when determining λ by formula (23).

That is why λ determined by relation (15) or (23) is called longitude in a broader sense. The corresponding value of longitude λ' from the interval $(-\pi,\pi)$ will be obtained as a remainder when dividing by 2π, i.e. by applying the formula

$$\lambda' = \lambda - 2\pi\,\mathrm{sgn}(\lambda)\left[\frac{|\lambda| + \pi}{2\pi}\right], \tag{25}$$

where $\mathrm{sgn}(\lambda)$ is equal to 1, 0 or −1, but according to whether λ is greater than, equal to or less than zero, and the square brackets indicate the largest integer function, i.e. $[x]$ is the largest integer that is less than x or equal to x (Figure 6).

If $\lambda \in (-\pi,\pi)$, then the longitude in a broader sense coincides with the usual longitude.

Numerical example 1
Given:
$\varphi_1 = 0°, \lambda_1 = 0°$
$\varphi_2 = 45°$
$\alpha = 45°$
Looking for: λ_2

Calculated:
Using expression (19), we calculate
$q_1 = 0, q_2 = 0.88137, \beta = 0, \lambda_2 = q_2$,

so by applying the formula (23), the value of longitude λ_2 in the broader sense in radians is 0.88137, i.e. in degrees 50°29'56''. As the calculated value is within the interval $(-\pi,\pi)$, the longitude coincides with the common longitude.

Numerical example 2
Given:
$\varphi_1 = 0°, \lambda_1 = 0°$
$\varphi_2 = 45°$
$\alpha = 80°$
Looking for: λ_2

Calculated:
Using expression (19), we calculate
$q_1 = 0, q_2 = 0.88137, \beta = 0, \lambda_2 = q_2 \tan 80° = 4.998518$.

By applying expression (23), the value of geographic longitude in the broader sense is calculated, which in radians is 4.99852, that is, in degrees 286°23'38''. Since the calculated value is not within the interval $(-\pi,\pi)$, the geographic longitude in the usual sense can be calculated using formula (26), which amounts to −1.28467 in radians, or −73°36'22'' in degrees.

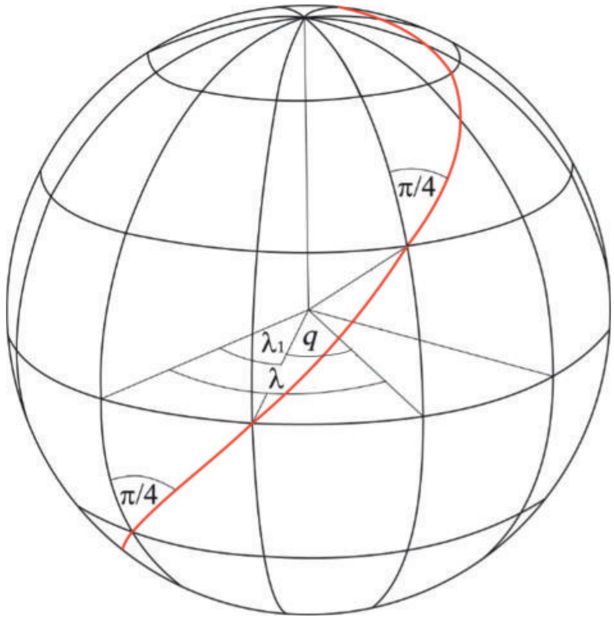


Fig. 7 Isometric latitude q as the difference of geographic longitudes.
Slika 7. Izometrijska širina q kao razlika geografskih dužina.

7 Special Cases

Meridians and parallels are special cases of loxodrome. For meridians $\alpha = k\pi$, $k \in Z$, Z is the set of all integers, and for parallels $\alpha = \pi/2 + k\pi$, $k \in Z$. Indeed, if we take $\alpha = k\pi$, $k \in Z$, then (14) turns into $s = R(\varphi - \varphi_1)$, and (15) into $\lambda = \lambda_1$.

For $\alpha = \pi/2 + k\pi$, $k \in Z$, (14) becomes $\varphi = \varphi_1$, and the differential equation $ds = R\cos\varphi_1 d\lambda$ gives the solution $s = R\cos\varphi_1(\lambda - \lambda_1)$.

8 Definition of Isometric Latitude Using Geographic Longitude

It is well known that latitude and longitude can also be defined using the corresponding angles related to the Earth's sphere or globe. Thus, latitude is the angle that the normal of the observed point closes with the equatorial plane. Longitude is the angle between the meridian passing through the observed point and the prime meridian. The question arises whether the isometric latitude can be interpreted in a similar way. Heck (1987)

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Grafički prikaz međusobne ovisnosti geografske i izometrijske širine dan je na slici 4.

Iz definicije izometrijske širine lako se izvedu ove relacije

$$\tanh q = \sin\varphi, \sinh q = \tan\varphi, \cosh q = \frac{1}{\cos\varphi},$$
$$\tanh\frac{q}{2} = \tan\frac{\varphi}{2}, \exp(q) = \tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right). \tag{21}$$

Nadalje, diferencijalna jednadžba (15) napisana uz pomoć izometrijske širine q postaje vrlo jednostavna i glasi

$$d\lambda = \tan\alpha\, dq. \tag{22}$$

Nakon integriranja dobijemo jednadžbu loksodrome na sferi u obliku

$$\lambda = q\tan\alpha + \beta, \tag{23}$$

gdje je β konstanta integracije. Ako želimo da loksodroma prolazi točkom s koordinatama (q_1,λ_1), potrebno je za integracijsku konstantu β uzeti

$$\beta = \lambda_1 - q_1\tan\alpha. \tag{24}$$

6. Poopćena geografska dužina λ

Uočimo da je za bilo koji $\varphi \in (-\pi/2,\pi/2)$, λ određen prema (16) realan broj koji općenito ne mora biti iz intervala $(-\pi,\pi)$ te nema značenje geografske dužine u uobičajenom smislu (slika 5). To se također jasno vidi pri određivanju λ po formuli (23).

Zbog toga λ određen relacijom (15) ili (23) zovemo *geografskom dužinom u širem smislu*. Odgovarajuća vrijednost geografske dužine λ' iz intervala $(-\pi,\pi)$ dobit će se kao ostatak pri dijeljenju s 2π, odnosno primjenom formule

$$\lambda' = \lambda - 2\pi\,\mathrm{sgn}(\lambda)\left[\frac{|\lambda|+\pi}{2\pi}\right], \tag{25}$$

gdje je $\mathrm{sgn}(\lambda)$ jednako 1, 0 ili −1, već prema tome je li λ veće, jednako ili manje od nule, a uglate zagrade označavaju funkciju najveće cijelo, tj. $[x]$ je najveći cijeli broj koji je manji od x ili jednak x (slika 6).

Ako je $\lambda \in (-\pi,\pi)$, onda se geografska dužina u širem smislu podudara s uobičajenom geografskom dužinom.

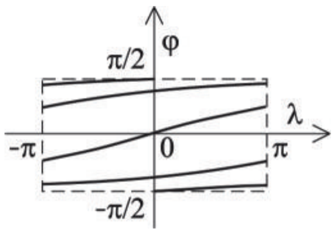
Numerički primjer 1

Zadano:

$\varphi_1 = 0°, \lambda_1 = 0°$

$\varphi_2 = 45°$

$\alpha = 45°$



Slika 6. Loksodroma u koordinatnom sustavu λ, φ, gdje je λ geografska dužina u uobičajenom smislu.

Fig. 6 Loxodrome in the coordinate system λ, φ, where λ is the longitude in the usual sense.

Traži se: λ_2

Izračunano:

Primjenom izraza (19) izračunamo

$q_1 = 0, q_2 = 0{,}88137, \beta = 0, \lambda_2 = q_2$,

pa je primjenom formule (23) vrijednost geografske dužine λ_2 u širem smislu u radijanima 0,88137, odnosno u stupnjevima 50°29'56''. Kako se izračunana vrijednost nalazi unutar intervala $(-\pi,\pi)$, geografska dužina u širem smislu podudara se s uobičajenom geografskom dužinom.

Numerički primjer 2

Zadano:

$\varphi_1 = 0°, \lambda_1 = 0°$

$\varphi_2 = 45°$

$\alpha = 80°$

Traži se: λ_2

Izračunano:

Primjenom izraza (19) izračunamo

$q_1 = 0, q_2 = 0{,}88137, \beta = 0, \lambda_2 = q_2 \tan 80° = 4{,}998518$.

Primjenom izraza (23) izračunana je vrijednost geografske dužine u širem smislu koja u radijanima iznosi 4,99852, odnosno u stupnjevima 286°23'38''. Kako se izračunana vrijednost ne nalazi unutar intervala $(-\pi,\pi)$, može se primjenom formule (25) izračunati geografska dužina u uobičajenom smislu, koja iznosi u radijanima −1,28467, odnosno u stupnjevima −73°36'22''.

7. Posebni slučajevi

Meridijani i paralele su specijalni slučajevi loksodrome. Za meridijane je $\alpha = k\pi$, $k\in Z$, Z je skup svih cijelih brojeva, a za paralele $\alpha = \pi/2 + k\pi$, $k\in Z$. Zaista, ako uzmemo $\alpha = k\pi$, $k\in Z$, tada (14) prelazi u $s = R(\varphi - \varphi_1)$, a (15) u $\lambda = \lambda_1$.

says in his famous monograph: "While the latitude φ can be given a geometric meaning of the direction of the normal on the surface, the numerical values of the isometric latitude cannot be interpret visually." We will show that it is not so, but that the isometric latitude can be interpret visually.

For this purpose, through any point with coordinates $(\varphi,\lambda) \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right) \times (-\pi,\pi)$ we draw a loxodrome on the sphere at an angle $\alpha = \frac{\pi}{4}$ (Figure 7).

The loxodrome equation according to (23) and (24) then reads

$$\lambda - \lambda_1 = q - q_1. \tag{26}$$

One nice property of such a loxodrome can be read from there: the difference in the isometric latitude of any two points on the loxodrome that closes the angle $\alpha = \pi/4$ on the sphere with meridians is equal to the difference in the geographic longitudes of those points. Geographical longitudes are understood in a broader sense.

The following is valid for the point where the loxodrome crosses the equator

$\varphi_1 = q_1 = 0.$

Now (23) can be written in the form

$q = \lambda - \lambda_1,$

which enables this definition of isometric latitude:

Definition. The isometric latitude of any point on the sphere is equal to the difference between the longitude (in a broader sense) of that point and the longitude of the point where the loxodrome drawn through the observed point at an angle $\alpha = \pi/4$ to the meridians (parallels) intersects the equator (Figure 7).

So, although isometric latitude contains the word "latitude" in its name and is usually formally defined using geographic latitude, it has just been explained how it can be very simply displayed and interpreted on a sphere as the difference of two longitudes. Analogous considerations apply to the loxodrome on the rotating ellipsoid.

9 Basic Tasks Along the Loxodrome on the Sphere

The first geodetic task for the loxodrome on a sphere of given radius R

Point T_1 (φ_1,λ_1) is given, the loxodrome that passes through that point closes the angle α to the meridians and the length of the arc is s. We are looking for point T_2 (φ_2,λ_2) on that loxodrome, which is distant from point T_1 along the loxodrome by s.

Solution:

According to expression (14) we have

$$\varphi_2 = \frac{s}{R}\cos\alpha + \varphi_1.$$

Then according to (19) we calculate

$$q_1 = \ln\tan\left(\frac{\pi}{4}+\frac{\varphi_1}{2}\right),\ q_2 = \ln\tan\left(\frac{\pi}{4}+\frac{\varphi_2}{2}\right).$$

According to (23) we have

$$\lambda_1 = q_1\tan\alpha + \beta,\ \lambda_2 = q_2\tan\alpha + \beta,$$

and from there

$$\lambda_2 = (q_2 - q_1)\tan\alpha + \lambda_1.$$

Numerical example:

Given:

Point T_1 (Zagreb): $\varphi_1 = 46°$ N, $\lambda_1 = 16°$ E

$\alpha = 158°$

$s = 420\,000$ m

$R = 6\,370\,000$ m

Looking for: φ_2 and λ_2

Calculated:

Point T_2 (Dubrovnik): $\varphi_2 = 42°30'$ N

$\lambda_2 = 18°$ E

The second geodetic problem for the loxodrome on a sphere of given radius R

The points T_1 (φ_1,λ_1) and T_2 (φ_2,λ_2) are given on the sphere of radius R. The angle to the meridians enclosed by the loxodrome connecting these two points is sought, and the length of the arc s of the loxodrome between these two points.

Solution:

First, we calculate according to (19).

$$q_1 = \ln\tan\left(\frac{\pi}{4}+\frac{\varphi_1}{2}\right),\ q_2 = \ln\tan\left(\frac{\pi}{4}+\frac{\varphi_2}{2}\right).$$

Then according to (23) we have

$$\lambda_1 = q_1\tan\alpha + \beta,\ \lambda_2 = q_2\tan\alpha + \beta,$$

And from there

$$\tan\alpha = \frac{\lambda_2 - \lambda_1}{q_2 - q_1}.$$

Finally, according to (14)

$$s = \frac{R(\varphi_2 - \varphi_1)}{\cos\alpha}.$$

Za $\alpha = \pi/2 + k\pi$, $k \in Z$, (14) prelazi u $\varphi = \varphi_1$, a diferencijalna jednadžba $ds = R\cos\varphi_1 d\lambda$ daje rješenje $s = R\cos\varphi_1(\lambda - \lambda_1)$.

8. Definicija izometrijske širine s pomoću geografske dužine

Dobro je poznato da se geografska širina i dužina mogu definirati i s pomoću odgovarajućih kutova vezanih uz Zemljinu sferu ili globus. Tako se geografskom širinom naziva kut koji normala promatrane točke zatvara s ekvatorskom ravninom. Geografska dužina je kut između meridijana što prolazi promatranom točkom i početnog meridijana. Postavlja se pitanje, može li se na sličan način interpretirati i izometrijska širina. Heck (1987) u svojoj poznatoj monografiji kaže: "Dok se geografskoj širini φ može dati geometrijsko značenje smjera normale na plohu, numeričke vrijednosti izometrijske širine ne mogu se zorno interpretirati." Mi ćemo pokazati da nije tako, nego da se izometrijska širina može zorno interpretirati.

U tu svrhu kroz bilo koju točku s koordinatama

$(\varphi,\lambda) \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right) \times (-\pi,\pi)$ povucimo na sferi loksodromu

pod kutem $\alpha = \frac{\pi}{4}$ (slika 7).

Jednadžba loksodrome prema (23) i (24) tada glasi

$$\lambda - \lambda_1 = q - q_1. \qquad (26)$$

Odatle se može pročitati jedno lijepo svojstvo takve loksodrome: razlika izometrijskih širina bilo kojih dviju točaka na loksodromi koja na sferi s meridijanima zatvara kut $\alpha = \pi/4$ jednaka je razlici geografskih dužina tih točaka. Pritom se podrazumijevaju geografske dužine u širem smislu.

Za točku u kojoj loksodroma presijeca ekvator vrijedi

$\varphi_1 = q_1 = 0$.

Sad se (23) može napisati u obliku

$q = \lambda - \lambda_1$,

što omogućava ovakvu definiciju izometrijske širine.

Definicija. Izometrijska širina bilo koje točke na sferi jednaka je razlici geografske dužine (u širem smislu) te točke i geografske dužine točke u kojoj loksodroma povučena kroz promatranu točku pod kutem $\alpha = \pi/4$ prema meridijanima (paralelama) siječe ekvator (slika 7).

Dakle, iako izometrijska širina u svom nazivu sadrži riječ "širina" i obično se formalno definira s pomoću geografske širine, upravo je objašnjeno kako se ona može vrlo jednostavno prikazati i interpretirati na sferi kao razlika dviju geografskih dužina. Analogna razmatranja vrijede i za loksodromu na rotacijskom elipsoidu.

9. Osnovni zadatci uzduž loksodrome na sferi

Prvi geodetski zadatak za loksodromu na sferi zadanog polumjera R

Zadana je točka $T_1(\varphi_1,\lambda_1)$, loksodroma koja prolazi tom točkom zatvara kut α prema meridijanima i duljina luka je *s*. Traži se točka $T_2(\varphi_2,\lambda_2)$ na toj loksodromi koja je od točke T_1 udaljena po loksodromi za *s*.

Rješenje:

Prema izrazu (14) imamo

$$\varphi_2 = \frac{s}{R}\cos\alpha + \varphi_1.$$

Zatim prema (19) računamo

$$q_1 = \ln\tan\left(\frac{\pi}{4} + \frac{\varphi_1}{2}\right),\ q_2 = \ln\tan\left(\frac{\pi}{4} + \frac{\varphi_2}{2}\right).$$

Prema (23) imamo

$$\lambda_1 = q_1 \tan\alpha + \beta,\ \lambda_2 = q_2 \tan\alpha + \beta,$$

i odatle

$$\lambda_2 = (q_2 - q_1)\tan\alpha + \lambda_1.$$

Numerički primjer:

Zadano:

Točka T_1 (Zagreb): $\varphi_1 = 46°$ N, $\lambda_1 = 16°$ E

$\alpha = 158°$

$s = 420\,000$ m

$R = 6\,370\,000$ m

Traži se: φ_2 i λ_2

Izračunano:

Točka T_2 (Dubrovnik): $\varphi_2 = 42°30'$ N

$\lambda_2 = 18°$ E

Drugi geodetski zadatak za loksodromu na sferi zadanog polumjera R

Zadane su točke $T_1(\varphi_1,\lambda_1)$ i $T_2(\varphi_2,\lambda_2)$ na sferi polumjera *R*. Traži se kut prema meridijanima koji zatvara loksodroma koja spaja te dvije točke, i duljina luka *s* te loksodrome između tih dviju točaka.

Rješenje:

Najprije prema (19) računamo

$$q_1 = \ln\tan\left(\frac{\pi}{4} + \frac{\varphi_1}{2}\right),\ q_2 = \ln\tan\left(\frac{\pi}{4} + \frac{\varphi_2}{2}\right).$$

Zatim prema (23) imamo

$$\lambda_1 = q_1 \tan\alpha + \beta,\ \lambda_2 = q_2 \tan\alpha + \beta,$$

Numerical example
Given:
Point T_1 (Zagreb): $\varphi_1 = 46°$ N, $\lambda_1 = 16°$ E
Point T_2 (Dubrovnik): $\varphi_2 = 42°30'$ N, $\lambda_2 = 18°$ E
$R = 6\,370\,000$ m
Looking for: α and *s*

Computed:
$\alpha = -22°15'03'' + k\cdot 180°$
Since $\varphi_1 > \varphi_2$ and $\lambda_1 < \lambda_2$, according to table 1 we choose that α for which $\alpha \in (\pi/2, \pi)$. Therefore, $\alpha = 157°44'56''$. After that, we get $s = 420$ km.

Remark
The question arises: how many loxodromes are there that connect the two points T_1 and T_2 on the sphere? The answer is: infinitely many, we just have to remember that instead of longitude (e.g. λ_2) it is possible to use longitude in the broader sense $\lambda_2 + 2k\pi$, $k \in Z$.

Numerical example
If the starting point T_1 is Zagreb with coordinates $\varphi_1 = 46°$ and $\lambda_1 = 16°$, and the end point T_2 is Dubrovnik with coordinates $\varphi_2 = 42°30'$, $\lambda_2 = 18° + k\,360°$ then for:
1. $k = 0$, $\lambda_2 = 18°$ calculated azimuth $\alpha = 157°44'56''$, distance $s = 420$ km
2. $k = 1$, $\lambda_2 = 378°$ calculated azimuth $\alpha = 90°46'25''$, distance $s = 28\,818$ km
3. $k = 2$, $\lambda_2 = 738°$ calculated azimuth $\alpha = 90°23'17''$, distance $s = 57\,473$ km
4. $k = 3$, $\lambda_2 = 1098°$ calculated azimuth $\alpha = 90°15'32''$, distance $s = 86\,129$ km
5. etc.

10 Loxodrome on a rotational ellipsoid

The following expressions define in geodesy and cartography a rotational ellipsoid with the center at the origin of the coordinate system, the semi-major axis *a* and the numerical eccentricity *e*:

$$\begin{aligned}&\vec{R}(\varphi,\lambda) = (x,y,z),\\ &x = x(\varphi,\lambda) = M\cos\varphi\cos\lambda,\ y = y(\varphi,\lambda) = M\cos\varphi\sin\lambda,\\ &z = z(\varphi,\lambda) = N(1-e^2)\sin\varphi\end{aligned} \tag{27}$$

$$(\varphi,\lambda) \in \Omega = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times [-\pi,\pi],\ (x,y,z) \in \mathrm{R}^3$$

We marked

$$M = M(\varphi) = \frac{a(1-e^2)}{\sqrt{(1-e^2\sin^2\varphi)^3}} \tag{28}$$

radius of curvature of the meridian and

$$N = N(\varphi) = \frac{a}{\sqrt{1-e^2\sin^2\varphi}} \tag{29}$$

radius of curvature of the intersection along the first vertical. The coefficients of the first differential form of this mapping are:

$$E = M^2,\ F = 0,\ G = N^2\cos^2\varphi.$$

Accordingly, the differential expressions (6)–(9), with agreement on the sign of the azimuth α according to Table 1, for any curve on the rotational ellipsoid are:

$$ds^2 = M^2\,d\varphi^2 + N^2\cos^2\varphi\,d\lambda^2, \tag{30}$$

$$\cos\alpha\,ds = M d\varphi, \tag{31}$$

$$\sin\alpha\,ds = N\cos\varphi\,d\lambda, \tag{32}$$

$$\tan\alpha = \frac{N\cos\varphi\,d\lambda}{M d\varphi}. \tag{33}$$

Let α = const. The differential equation of the loxodrome on the rotating ellipsoid is then e.g. (31), and can be solved as follows:

$$\cos\alpha \int ds = \int M d\varphi,$$

which after integration gives

$$s\cos\alpha = s_m(\varphi) - s_m(\varphi_1), \tag{34}$$

where

$$s_m(\varphi) = \int_0^{\varphi} M d\varphi, \tag{35}$$

so (34) is the equation of the loxodrome connecting the latitude φ and the arc length *s*. This loxodrome passes through a point with latitude φ_1 and at that point the arc length is 0.

Unlike the derivation of the loxodrome equation on the sphere, the integral on the right in (35) is an elliptic integral that appears when calculating the length of the arc of the meridian on the rotational ellipsoid, and which cannot be directly integrated, but instead, series or some other mathematical methods are applied. Lapaine (1990, 1994) showed that for calculating the length of the arc of the meridian from the equator to the geodetic latitude φ, a suitable formula is the following

$$s_m(\varphi) = A[\varphi + \sin 2\varphi(c_1 + (c_2 + (c_3 + (c_4 + c_5\cos 2\varphi)\cos 2\varphi)\cos 2\varphi)\cos 2\varphi)] + \cdots \tag{36}$$

where $A, c_1, c_2, \ldots, c_5$ are the corresponding coefficients.

i odatle

$$\tan\alpha = \frac{\lambda_2 - \lambda_1}{q_2 - q_1}.$$

Konačno prema (14)

$$s = \frac{R(\varphi_2 - \varphi_1)}{\cos\alpha}.$$

Numerički primjer

Zadano:

Točka T_1 (Zagreb): $\varphi_1 = 46°$ N, $\lambda_1 = 16°$ E

Točka T_2 (Dubrovnik): $\varphi_2 = 42°30'$ N, $\lambda_2 = 18°$ E

$R = 6\ 370\ 000$ m

Traži se: α i s

Izračunano:

$\alpha = -22°15'03'' + k \cdot 180°$

Budući da je $\varphi_1 > \varphi_2$ i $\lambda_1 < \lambda_2$, to prema tablici 1 biramo onaj α za koji je $\alpha \in (\pi/2,\pi)$. Dakle, α = 157°44'56". Nakon toga dobijemo s = 420 km.

Napomena

Postavlja se pitanje: koliko ima loksodroma koje spajaju dvije točke T_1 i T_2 na sferi? Odgovor je: beskonačno mnogo, samo se treba sjetiti da je umjesto geografske dužine (npr. λ_2) moguće upotrijebiti geografsku dužinu u širem smislu $\lambda_2 + 2k\pi$, $k \in Z$.

Numerički primjer

Ako je početna točka T_1 Zagreb s koordinatama φ_1 = 46° i λ_1 = 16°, a krajnja točka T_2 Dubrovnik s koordinatama φ_2 = 42°30', $\lambda_2 = 18° + k\,360°$ tada je za:

1. $k = 0$, $\lambda_2 = 18°$ izračunani azimut α = 157°44'56'', duljina s = 420 km
2. $k = 1$, $\lambda_2 = 378°$ izračunani azimut α = 90°46'25'', duljina s = 28 818 km
3. $k = 2$, $\lambda_2 = 738°$ izračunani azimut α = 90°23'17'', duljina s = 57 473 km
4. $k = 3$, $\lambda_2 = 1098°$ izračunani azimut α = 90°15'32'', duljina s = 86 129 km
5. itd.

10. Loksodroma na rotacijskom elipsoidu

Sljedeći izrazi definiraju u geodeziji i kartografiji rotacijski elipsoid sa središtem u ishodištu koordinatnog sustava, velikom poluosi a i numeričkim ekscentricitetom e:

$$\begin{aligned} &\vec{R}(\varphi,\lambda) = (x,y,z),\\ &x = x(\varphi,\lambda) = M\cos\varphi\cos\lambda,\ y = y(\varphi,\lambda) = M\cos\varphi\sin\lambda,\\ &z = z(\varphi,\lambda) = N(1 - e^2)\sin\varphi \end{aligned} \tag{27}$$

$$(\varphi,\lambda) \in \Omega = \left[-\frac{\pi}{2},\frac{\pi}{2}\right] \times [-\pi,\pi],\ (x,y,z) \in \mathrm{R}^3$$

Pri tome su

$$M = M(\varphi) = \frac{a(1-e^2)}{\sqrt{(1-e^2\sin^2\varphi)^3}} \tag{28}$$

polumjer zakrivljenosti meridijana i

$$N = N(\varphi) = \frac{a}{\sqrt{1-e^2\sin^2\varphi}} \tag{29}$$

polumjer zakrivljenosti presjeka po prvom vertikalu. Koeficijenti prve diferencijalne forme toga preslikavanja su:

$$E = M^2,\ F = 0,\ G = N^2\cos^2\varphi.$$

Prema tome diferencijalni izrazi (6)–(9), uz dogovor po predznaku azimuta α prema tablici 1, za bilo koju krivulju na rotacijskom elipsoidu su:

$$ds^2 = M^2 d\varphi^2 + N^2\cos^2\varphi\, d\lambda^2, \tag{30}$$

$$\cos\alpha\, ds = M d\varphi \tag{31}$$

$$\sin\alpha\, ds = N\cos\varphi\, d\lambda, \tag{32}$$

$$\tan\alpha = \frac{N\cos\varphi\, d\lambda}{M d\varphi}. \tag{33}$$

Neka je α = *const*. Diferencijalna jednadžba loksodrome na rotacijskom elipsoidu je tada npr. (31), a može se riješiti na sljedeći način:

$$\cos\alpha \int ds = \int M d\varphi,$$

što nakon integracije daje

$$s\cos\alpha = s_m(\varphi) - s_m(\varphi_1), \tag{34}$$

gdje je

$$s_m(\varphi) = \int_0^\varphi M d\varphi, \tag{35}$$

pa je (34) jednadžba loksodrome koja povezuje geodetsku širinu φ i duljinu luka s. Ta loksodroma prolazi točkom s geodetskom širinom φ_1 i u toj je točki duljina luka 0.

Za razliku od izvoda jednadžbe loksodrome na sferi, integral s desna strane u (35) je eliptički integral koji se pojavljuje pri računanju duljine luka meridijana na rotacijskom elipsoidu, i kojega nije moguće neposredno integrirati već se primijenjuju razvoji u redove ili neke druge matematičke metode. Lapaine (1990, 1994) je pokazao da je za računanje duljine luka meridijana od ekvatora do geodetske širine φ pogodna formula

$$s_m(\varphi) = A[\varphi + \sin 2\varphi(c_1 + (c_2 + (c_3 + (c_4 + c_5\cos 2\varphi)\cos 2\varphi)\cos 2\varphi)\cos 2\varphi)] + \cdots \tag{36}$$

gdje su $A, c_1, c_2, \ldots, c_5$ odgovarajući koeficijenti.

In formula (36), the length of the arc of the loxodrome is expressed as a function of geodetic latitude. If it is necessary to express the geodetic latitude as a function of the length of the arc of the loxodrome, then we can use the formula that determines the geodetic latitude of a point on the meridian for which the length of the arc of the meridian from the equator to that point is known Lapaine (1990, 1994):

$$\varphi(s_m) = \psi + \sin 2\psi(c_1 + (c_2 + (c_3 + (c_4 + c_5 \cos 2\psi)\cos 2\psi)\cos 2\psi)\cos 2\psi) + \cdots \tag{37}$$

where $c_1, c_2, ..., c_5$ are corresponding coefficients, and

$$\psi = \frac{s_m(\varphi)}{A}, \; s_m(\varphi) = s\cos\alpha + s_m(\varphi_1). \tag{38}$$

Loxodromes on a rotational ellipsoid, like on a sphere, are generally spiral curves that wrap around each pole an infinite number of times, and never reach it, although their length is finite. The length of the loxodrome from pole to pole is equal to the length of the arc of the meridian divided by the cosine of the angle α. Indeed, in formula (34) we should put $\varphi_1 = -\frac{\pi}{2}$, $\varphi = \frac{\pi}{2}$, so we get

$$s = \frac{1}{\cos\alpha}\left[s_m\left(\frac{\pi}{2}\right) - s_m\left(-\frac{\pi}{2}\right)\right] = \frac{2}{\cos\alpha} s_m\left(\frac{\pi}{2}\right),$$

$$\alpha \neq \frac{\pi}{2} + k\pi, k \in Z.$$

If we start with the differential equation (32) we cannot integrate it immediately, but we should first express φ s by means of λ or s. Therefore, we prefer to take equation (33) which can be integrated if we write it in the form

$$d\lambda = \tan\alpha \frac{M d\varphi}{N\cos\varphi}. \tag{39}$$

After integration we get

$$\lambda = \tan\alpha\left[\ln\tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)\left(\frac{1-e\sin\varphi}{1+e\sin\varphi}\right)^{\frac{e}{2}}\right] + \beta = \tan\alpha\left[\tanh^{-1}(\sin\varphi) - e\tanh^{-1}(e\sin\varphi)\right] + \beta. \tag{40}$$

If we want the loxodrome to pass through the point with geodetic coordinates (φ_1, λ_1), it is necessary to take the integration constant β

$$\beta = \lambda_1 - \tan\alpha\left[\ln\tan\left(\frac{\pi}{4}+\frac{\varphi_1}{2}\right)\left(\frac{1-e\sin\varphi_1}{1+e\sin\varphi_1}\right)^{\frac{e}{2}}\right] = \lambda_1 - \tan\alpha\left[\tanh^{-1}(\sin\varphi_1) - e\tanh^{-1}(e\sin\varphi_1)\right]. \tag{41}$$

Finally, if we want a connection between the longitude λ and the arc length of the loxodrome s, it is necessary to include φ from (40) including (41), (42) and (43) in (45).

11 Isometric Latitude and Loxodrome on a Rotational Ellipsoid

Taught by the experience from the previous sectionis on isometric latitude and loxodromes on a sphere, let us try an analogous approach on a rotational ellipsoid. The isometric latitude q in the theory of map projections is defined by means of the geodetic latitude φ and the differential equation

$$dq = \frac{M d\varphi}{N\cos\varphi}. \tag{42}$$

The solution of differential equation (42) is

$$q = \ln\tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)\left(\frac{1-e\sin\varphi}{1+e\sin\varphi}\right)^{\frac{e}{2}} = \tanh^{-1}(\sin\varphi) - e\tanh^{-1}(e\sin\varphi). \tag{43}$$

with the assumption that for the integration constant we took the value that gives $q = 0$ for $\varphi = 0$. The inverse function cannot be written in a finite form using elementary functions, but different approximation procedures or approximate formulas are used (Frančula 2004).

If the conformal latitude χ is introduced as follows

$$\tan\left(\frac{\pi}{4}+\frac{\chi}{2}\right) = \tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)\left(\frac{1-e\sin\varphi}{1+e\sin\varphi}\right)^{\frac{e}{2}}, \tag{44}$$

then between it and the isometric latitude q, analogous to relations (21), the following relations apply:

$$\tanh q = \sin\chi, \; \sinh q = \tan\chi, \; \cosh q = \frac{1}{\cos\chi}, \quad \tanh\frac{q}{2} = \tan\frac{\chi}{2}, \; \exp(q) = \tan\left(\frac{\pi}{4}+\frac{\chi}{2}\right). \tag{45}$$

Furthermore, the differential equation (39) written with the help of the isometric latitude q becomes very simple and formally reads as in (22), it is only necessary to note that here we are talking about the isometric latitude q on the rotational ellipsoid defined by (43). After integration, we get the equation of the loxodrome on the rotational ellipsoid in the form (23). If we want the rhombus to pass through the point with geographic coordinates (q_1, λ_1), it is necessary to take the integration constant β as in (24).

U formuli (36) izražena je duljina luka loksodorme kao funkcija geodetske širine. Ako je potrebno izraziti geodetsku širinu kao funkciju duljine luka loksodrome, tada možemo upotrijebiti formulu s pomoću koje se određuje geodetska širina točke na meridijanu za koju je poznata duljina luka meridijana od ekvatora do te točke Lapaine (1990, 1994):

$$\varphi(s_m) = \psi + \sin 2\psi(c_1 + (c_2 + (c_3 + (c_4 + c_5\cos 2\psi)\cos 2\psi)\cos 2\psi)\cos 2\psi) + \cdots \tag{37}$$

gdje su $c_1, c_2, ..., c_5$ odgovarajući koeficijenti, a

$$\psi = \frac{s_m(\varphi)}{A},\ s_m(\varphi) = s\cos\alpha + s_m(\varphi_1). \tag{38}$$

Loksodrome na rotacijskom elipsoidu su, kao i na sferi, općenito spiralne krivulje koje se omataju oko svakog pola beskonačno mnogo puta, i do njega nikad ne stižu, premda je njihova duljina konačna. Duljina loksodrome od pola do pola jednaka je duljini luka meridijana podijeljenog kosinusom kuta α.

Zaista, u formulu (34) treba staviti $\varphi_1 = -\frac{\pi}{2}$, $\varphi = \frac{\pi}{2}$, pa se dobije

$$s = \frac{1}{\cos\alpha}\left[s_m\left(\frac{\pi}{2}\right) - s_m\left(-\frac{\pi}{2}\right)\right] = \frac{2}{\cos\alpha}s_m\left(\frac{\pi}{2}\right),$$

$$\alpha \neq \frac{\pi}{2} + k\pi,\ k \in Z.$$

Ako krenemo s diferencijalnom jednadžbom (32) ne možemo je odmah integrirati, nego bi najprije trebalo izraziti φ s pomoću λ ili s. Stoga ćemo radije uzeti jednadžbu (33) koja se može integrirati ako je napišemo u obliku

$$d\lambda = \tan\alpha\frac{Md\varphi}{N\cos\varphi}. \tag{39}$$

Nakon integriranja dobijemo

$$\lambda = \tan\alpha\left[\ln\tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)\left(\frac{1-e\sin\varphi}{1+e\sin\varphi}\right)^{\frac{e}{2}}\right] + \beta = \tan\alpha\left[\tanh^{-1}(\sin\varphi) - e\tanh^{-1}(e\sin\varphi)\right] + \beta. \tag{40}$$

Ako želimo da loksodroma prolazi točkom s geodetskim koordinatama (φ_1, λ_1), potrebno je za integracijsku konstantu β uzeti

$$\beta = \lambda_1 - \tan\alpha\left[\ln\tan\left(\frac{\pi}{4}+\frac{\varphi_1}{2}\right)\left(\frac{1-e\sin\varphi_1}{1+e\sin\varphi_1}\right)^{\frac{e}{2}}\right] = \lambda_1 - \tan\alpha\left[\tanh^{-1}(\sin\varphi_1) - e\tanh^{-1}(e\sin\varphi_1)\right]. \tag{41}$$

Konačno, ako želimo vezu između geodetske dužine λ i duljine luka loksodrome s potrebno je φ iz (40) uključujući (41), (42) i (43) uvrstiti u (45).

11. Izometrijska širina i loksodroma na rotacijskom elipsoidu

Poučeni iskustvom iz prehodnih poglavlja o izometrijskoj širini i loksodromi na sferi, pokušajmo s analognim pristupom na rotacijskom elipsoidu. Izometrijsku širinu q u teoriji kartografskih projekcija definiramo s pomoću geodetske širine φ i diferencijalne jednadžbe

$$dq = \frac{Md\varphi}{N\cos\varphi}. \tag{42}$$

Rješenje diferencijalne jednadžbe (42) je

$$q = \ln\tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)\left(\frac{1-e\sin\varphi}{1+e\sin\varphi}\right)^{\frac{e}{2}} = \tanh^{-1}(\sin\varphi) - e\tanh^{-1}(e\sin\varphi). \tag{43}$$

uz pretpostavku da smo za integracijsku konstantu uzeli onu vrijednost koja za $\varphi = 0$ daje $q = 0$. Obratna veza, odnosno inverzna funkcija ne može se napisati u konačnom obliku s pomoću elementarnih funkcija, nego se koriste različiti aproksimacijski postupci ili približne formule (Frančula 2004).

Ako se uvede konformna širina χ na sljedeći način

$$\tan\left(\frac{\pi}{4}+\frac{\chi}{2}\right) = \tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)\left(\frac{1-e\sin\varphi}{1+e\sin\varphi}\right)^{\frac{e}{2}}, \tag{44}$$

onda između nje i izometrijske širine q vrijede, analogno relacijama (21), ove relacije:

$$\tanh q = \sin\chi,\ \sinh q = \tan\chi,\ \cosh q = \frac{1}{\cos\chi},$$
$$\tanh\frac{q}{2} = \tan\frac{\chi}{2},\ \exp(q) = \tan\left(\frac{\pi}{4}+\frac{\chi}{2}\right). \tag{45}$$

Nadalje, diferencijalna jednadžba (39) napisana uz pomoć izometrijske širine q postaje vrlo jednostavna i formalno glasi kao u (22), samo treba pripaziti da je ovdje riječ o izometrijskoj širini q na rotacijskom elipsoidu definiranoj s (43). Nakon integriranja dobijemo jednadžbu loksodrome na rotacijskom elipsoidu u obliku (23). Ako želimo da loksodroma prolazi točkom s geografskim koordinatama (q_1, λ_1), potrebno je za integracijsku konstantu β uzeti kao u (24).

12. Osnovni zadaci uzduž loksodrome na rotacijskom elipsoidu

Prvi geodetski zadatak za loksodromu na rotacijskom elipsoidu

Zadana je točka $T_1(\varphi_1, \lambda_1)$na rotacijskom elipsoidu s poluosima a i b, loksodroma koja prolazi tom točkom zatvara kut α prema meridijanima i duljina luka je s.

12 Basic Tasks Along the Loxodrome on the Rotational Ellipsoid

The first geodetic problem for a loxodrome on a rotational ellipsoid

The point $T_1(\varphi_1,\lambda_1)$ on the rotational ellipsoid with semi-axes a and b is given, the loxodrome that passes through that point closes the angle α to the meridians and the length of the arc is s. The point $T_2(\varphi_2,\lambda_2)$ is sought on that loxodrome which is far from the point T_1 along the loxodrome for s.

Solution:

According to formula (43) we can write

$$s_m(\varphi_2) = s\cos\alpha + s_m(\varphi_1).$$

Then according to formulas (40)-(42) we calculate

$$\varphi_2 = \varphi_2(\psi),$$

where $\psi = \dfrac{s_m(\varphi_2)}{A}$.

Now, according to formula (43) we can calculate

$$q_1 = \tanh^{-1}(\sin\varphi_1) - e\tanh^{-1}(e\sin\varphi_1),$$
$$q_2 = \tanh^{-1}(\sin\varphi_2) - e\tanh^{-1}(e\sin\varphi_2),$$

and finally based on formulas (45)

$$\lambda_2 = (q_2 - q_1)\tan\alpha + \lambda_1.$$

The second geodetic problem for a loxodrome on a rotational ellipsoid

Points $T_1(\varphi_1,\lambda_1)$ and $T_2(\varphi_2,\lambda_2)$ are given on the rotational ellipsoid with semi-axes a and b. The angle α to the meridians enclosed by the loxodrome connecting these two points is sought, and the length s of the arc from the loxodrome between these two points.

Solution:

First, we calculate using formula (43)

$$q_1 = \tanh^{-1}(\sin\varphi_1) - e\tanh^{-1}(e\sin\varphi_1),$$
$$q_2 = \tanh^{-1}(\sin\varphi_2) - e\tanh^{-1}(e\sin\varphi_2).$$

Then we have

$$\tan\alpha = \frac{\lambda_2 - \lambda_1}{q_2 - q_1}$$

and finally $s = \dfrac{s_m(\varphi_2) - s_m(\varphi_1)}{\cos\alpha}$.

13 Conclusion

In this paper, the close connection between isometric latitude and loxodrome is explained in detail. A new generalized definition of the loxodrome on any surface is proposed. Its differential equations are derived, which are solved on a sphere and on a rotational ellipsoid. A new concept of generalized geographic longitude was introduced, which appears in a natural way when solving the differential equation of the loxodrome. Generalized longitude allows defining isometric latitude in a new way. The relations between the isometric latitude and the geographic longitude on the sphere and geodetic longitude on the ellipsoid, are derived. At the end, basic geodetic tasks are solved along the loxodrome on the sphere and on the rotational ellipsoid.

Traži se točka $T_2(\varphi_2,\lambda_2)$ na toj loksodromi koja je od točke T_1 udaljena po loksodromi za s.

Rješenje:
Prema formuli (43) možemo napisati

$$s_m(\varphi_2) = s\cos\alpha + s_m(\varphi_1).$$

Zatim prema formulama (40)-(42) izračunamo

$$\varphi_2 = \varphi_2(\psi),$$

gdje je $\psi = \dfrac{s_m(\varphi_2)}{A}$.

Sad možemo prema formuli (43) izračunati

$$q_1 = \tanh^{-1}(\sin\varphi_1) - e\tanh^{-1}(e\sin\varphi_1),$$
$$q_2 = \tanh^{-1}(\sin\varphi_2) - e\tanh^{-1}(e\sin\varphi_2),$$

i na kraju na temelju formula (45)

$$\lambda_2 = (q_2 - q_1)\tan\alpha + \lambda_1.$$

Drugi geodetski zadatak za loksodromu na rotacijskom elipsoidu

Zadane su točke $T_1(\varphi_1,\lambda_1)$ i $T_2(\varphi_2,\lambda_2)$ na rotacijskom elipsoidu s poluosima a i b. Traži se kut α prema meridijanima koji zatvara loksodroma koja spaja te dvije točke, i duljina luka s loksodrome između tih dviju točaka.

Rješenje:
Najprije prema formuli (43) izračunamo

$$q_1 = \tanh^{-1}(\sin\varphi_1) - e\tanh^{-1}(e\sin\varphi_1),$$
$$q_2 = \tanh^{-1}(\sin\varphi_2) - e\tanh^{-1}(e\sin\varphi_2).$$

Zatim imamo

$$\tan\alpha = \frac{\lambda_2 - \lambda_1}{q_2 - q_1}$$

i konačno $s = \dfrac{s_m(\varphi_2) - s_m(\varphi_1)}{\cos\alpha}$.

13. Zaključak

U ovom je radu detaljno obrazložena uska povezanost izometrijske širine i loksodrome. Predložena je nova poopćena definicija loksodrome na bilo kojoj plohi. Izvedene su njezine diferencijalne jednaždbe koje se rješavaju na sferi i na rotacijskom elipsoidu. Uveden je novi pojam poopćene geografske dužine, koja se pojavljuje na prirodan način pri rješavanju diferencijalne jednadžbe lokosodrome. Poopćena geografska dužina omogućuje definiranje izometrijske širine na nov način. Izvode se relacije između izometrijske širine i geografske dužine na sferi i geodetske dužine na elipsoidu. Na kraju se rješavaju osnovni geodetski zadatci uzduž loksodrome na sferi i na rotacijskom elipsoidu.

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