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Cylindrical Projections as a Limiting Case of Conic Projections

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Abstract. Lambert (1772) derived the equation of the Mercator projection as a limiting case of a conformal conic projection. In this paper, we give a derivation for equidistant, equal-area, conformal and perspective cylindrical projections as limiting cases of equidistant, equal-area, conformal and perspective conic projections. In this article the conic and cylindrical projections are not projections on a cone or a cylinder whose surfaces are cut and developed into a plane, but rather mappings of the sphere directly into the plane. Exceptions are projections that are defined as mappings on the surface of a cone or plane, as is the case with perspective projections. In the end, we prove that it is not always possible to obtain a corresponding cylindrical projection as a limiting case from a conic projection, as one might conclude at first glance. Therefore, the final conclusion is that it is not advisable to interpret cylindrical projections as limiting cases of conic projections.

Keywords: conic projection, cylindrical projection, J. H. Lambert

1 Introduction

In books and textbooks on map projections, cylindrical, conic, and azimuthal projections are usually considered separately. It is sometimes mentioned that cylindrical and azimuthal projections can be interpreted as limiting cases of conic ones (Lee 1944, Kavrayskiy 1958, 1959, Jovanović 1983, Vakhrameyeva et al. 1986, Snyder 1987, Kuntz 1990, Canters 2022, Monmonier 2004, Serapinas 2005), but there are only a few attempts to prove it (Hinks 1912, Hoschek 1969, Daners 2012).

Kimerling (2010) in his blog introduced a way of thinking about similarities among projections – that seemingly distinct projections may actually be parts of a continuum of projections created by varying the parameters of a single pair of projection equations transforming the latitude and longitude coordinates on the sphere or ellipsoid into Cartesian coordinates on a flat projection surface. Such projection continuums are illustrated by animations that show the changes in the graticule and coastline as a certain projection parameter is varied systematically through a large range of values. Kimerling also provides animations showing the transition from conic to cylindrical or azimuthal projections.

In this paper, we will supplement the explanations and derivations of Hinks (1912), who applies Lambert's method of derivation (1772), although he does not mention Lambert.

The Lambert conformal conic projection is one of the most famous map projections. It is still used today in many countries. This projection was proposed by Lambert in his *Notes and Supplements to the Establishment of Earth and Sky Maps* published 250

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Cilindrične projekcije kao granični slučaj konusnih projekcija

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Sažetak. Lambert (1772) je izveo jednadžbu Mercatorove projekcije kao graničnog slučaja konformne konusne projekcije. U ovom radu dajemo izvod za ekvidistantnu, ekvivalentnu, konformnu i perspektivnu cilindričnu projekciju kao granične slučajeve ekvidistantne, ekvivalentne, konformne i perspektivne konusne projekcije. U ovom članku konusne i cilindrične projekcije nisu projekcije na stožac ili valjak čije su plohe izrezane i razvijene u ravninu, već preslikavanje sfere izravno u ravninu. Iznimke su projekcije koje su definirane kao preslikavanja na plohu stošca ili ravnine, kao što je slučaj s perspektivnim projekcijama. Na kraju dokazujemo da nije uvijek moguće iz konusne projekcije dobiti odgovarajuću cilindričnu projekciju kao granični slučaj, kako bi se dalo zaključiti na prvi pogled. Stoga je konačni zaključak da cilindrične projekcije nije uputno tumačiti kao granične slučajeve konusnih projekcija.

Ključne riječi: konusna projekcija, cilindrična projekcija, J. H. Lambert

1. Uvod

U knjigama i udžbenicima o kartografskim projekcijama obično se odvojeno razmatraju cilindrične, konusne i azimutne projekcije. Ponekad se spominje da se cilindrične i azimutne projekcije može tumačiti kao granični slučaj konusnih projekcija (Lee 1944, Kavrayskiy 1958, 1959, Jovanović 1983, Vakhrameyeva et al. 1986, Snyders 1987, Kuntz 1990, Canters 2022, Monmonier 2004, Serapinas 2005), a postoji samo nekoliko pokušaja da se to dokaže (Hinks 1912, Hoschek 1969, Daners 2012).

Kimerling (2010) je u svojem blogu predstavio način razmišljanja o sličnostima među projekcijama – naizgled različite projekcije mogu zapravo biti dijelovi kontinuuma projekcija stvorenih variranjem parametara jednog para jednadžbi projekcija transformirajući koordinate geografske širine i dužine na sferi ili elipsoidu u kartezijeve koordinate u ravnini. Takvi su projekcijski kontinuumi ilustrirani animacijama koje pokazuju promjene u mreži i obalnoj crti, dok se određeni projekcijski parametar mijenja kroz veliki raspon vrijednosti. Kimerling također nudi animacije koje prikazuju prijelaz s konusne na cilindrične ili azimutne projekcije.

U ovom ćemo članku dopuniti objašnjenja i izvode Hinksa (1912) koji, iako Lamberta ne spominje, primjenjuje Lambertovu metodu (1772).

Lambertova je konformna konusna projekcija jedna od najpoznatijih kartografskih projekcija. I danas se koristi u mnogim zemljama. Tu je projekciju predložio Lambert u svojim *Bilješkama i dodatcima za uspostavljanje karata Zemlje i neba* objavljenima prije 250 godina. Četvrti je pododjeljak *Općenitija metoda predstavljanja sferne površine tako da svi kutovi zadrže svoje veličine*. Taj je pododjeljak dalje podijeljen i počinje s §47 u kojem Lambert piše da

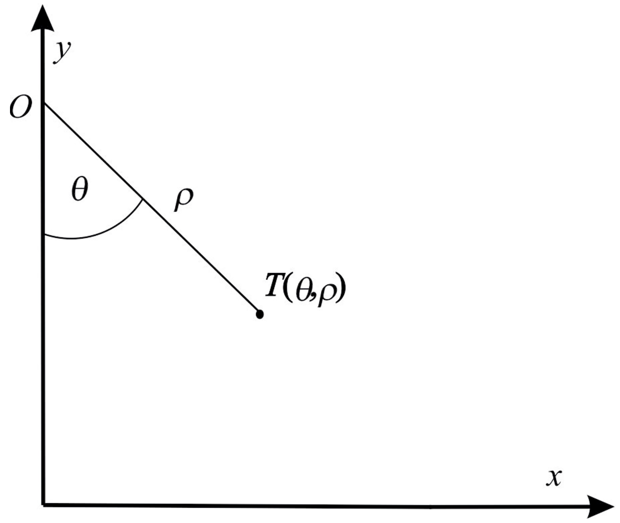


Fig. 1 Point *T* and its polar coordinates θ and ρ. The origin *O* of the polar coordinate system is usually located on the *y*-axis of the rectangular coordinate system in the projection plane.
Slika 1. Točka *T* i njezine polarne koordinate θ i ρ. Ishodište *O* polarnog koordinatnog sustava obično je smješteno na osi *y* pravokutnog koordinatnog sustava u ravnini projekcije.

years ago. The fourth subsection is *A more general method of representing a spherical surface so that all angles preserve their sizes*. That subsection is further divided and begins with §47 in which Lambert writes that the stereographic representation of the spherical surface, as well as Mercator nautical charts, have the property that all angles retain the size they had on the surface of the globe. In §48 Lambert derives the formula for the conformal projection of the unit sphere. After that he derived the Mercator projection as a limiting case of his conformal conic projection.

Hinks (1912) uses the same method as Lambert for the construction of a cylindrical projection as a limiting case of a conic projection. From the conic equidistant along the meridian (simple conic) he derives the cylindrical equidistant projection (in French *projection plate carrée*, in German *quadratische Plattkarte*). From the conformal conic (conic orthomorphic) he derives the conformal cylindrical projection (cylindrical orthomorphic – Mercator). For a simple equal-area projection with one standard parallel he does not give a derivation.

In this paper, we also give a derivation for the equal-area cylindrical projection as a limiting case of the equal-area conic projection. In addition, we give a derivation for the central perspective cylindrical projection as a limiting case of the central conic perspective projection.

It should be emphasized that Hinks defines a standard parallel as a parallel of true length: "One parallel, and sometimes a second, is made of the true length; that is to say, if the map is to be on the scale of one-millionth, the length of the complete parallel on the map will be one-millionth of the corresponding terrestrial parallel. This is called a Standard parallel." This definition does not correspond to today's understanding of distortions, according to which one should distinguish between standard parallels, equidistantly mapped parallels and parallels that have preserved their length in mapping.

Furthermore, Hinks implies in his derivations that a conic projection is a projection on a cone and that a cylindrical projection is a projection on a cylinder. In this paper, the conical and cylindrical projections are not projections on a cone or a cylinder whose surfaces are cut and developed into a plane, but rather mappings of the sphere directly into the plane. Exceptions are projections that are defined as mappings on the surface of a cone, as is the case with perspective projections.

Finally, we prove that it is not always possible to obtain a corresponding cylindrical projection from a conic projection, as one might conclude at first glance. Although very simple, it is the most important contribution of this article.

The equations of normal aspect conic projections are usually given in the polar coordinate system in the plane of the projection:

$$\theta = m\lambda,\ \ \rho = \rho(\varphi), \qquad (1)$$

where $\varphi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\lambda \in (-\pi, \pi)$ are the latitude and longitude, respectively, m the parameter, $0 < m \leq 1$, θ and ρ the polar coordinates in the plane of projection (Figure 1).

A question arises: from a normal aspect conic projection given by the equations in the polar coordinate system in the projection plane (1), how can we, as a special limiting case derive the equations of the normal aspect cylindrical projection in the rectangular system in the projection plane

$$x = n\lambda,\ \ y = y(\varphi) \qquad (2)$$

which will have analogous properties as the conic projection given by the equations (1). We are looking for a cylindrical projection (2), which is the counterpart of the conic one (1). For example, if the equal-area conic projection is given by the equations (1), stereografski prikaz sferne površine, kao i Mercatorove pomorske karte, imaju svojstvo da svi kutovi zadržavaju veličinu koju su imali na površini globusa. U §48 Lambert izvodi formulu za konformnu projekciju jedinične sfere. Nakon toga je izveo Mercatorovu projekciju kao granični slučaj njegove konformne konusne projekcije.

Hinks (1912) se koristi istom metodom kao Lambert za konstrukciju cilindrične projekcije kao graničnog slučaja konusne projekcije. Iz konusne ekvidistantne projekcije uzduž meridijana (jednostavne konusne) on izvodi cilindričnu ekvidistantnu projekciju (na francuskom *projection plate carrée*, na njemačkom *quadratische Plattkarte*). Iz konformne konusne projekcije (*conic orthomorphic*) izvodi konformnu cilindričnu (*cylindrical orthomorphic* – Mercator). Za jednostavnu ekvivalentnu projekciju s jednom standardnom paralelom Hinks ne daje izvod.

U ovom članku dajemo i izvod za ekvivalentnu cilindričnu projekciju kao granični slučaj ekvivalentne konusne projekcije. Dodatno, dajemo izvod za centralnu perspektivnu cilindričnu projekciju kao granični slučaj centralne konusne perspektivne projekcije.

Treba naglasiti da Hinks standardnu paralelu definira kao paralelu stvarne duljine: "Jedna paralela, a ponekad i druga, ima pravu duljinu; to jest, ako karta treba biti u mjerilu jedan prema milijun, duljina cijele paralele na karti bit će milijunti dio odgovarajuće zemaljske paralele. To se zove standardna paralela." Ta definicija ne odgovara današnjem shvaćanju distorzija prema kojem treba razlikovati standardne paralele, ekvidistantno preslikane paralele i paralele koje su u preslikavanju sačuvale svoju duljinu.

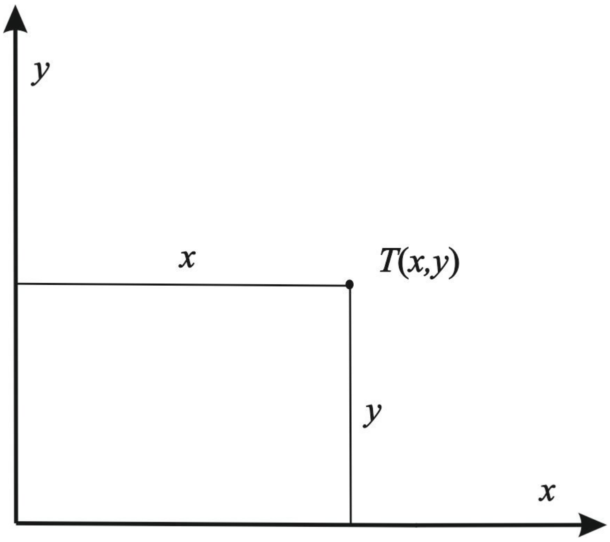
Nadalje, Hinks u svojim izvodima implicira da je konusna projekcija projekcija na stožac, a da je cilindrična projekcija projekcija na valjak. U ovom radu konusne i cilindrične projekcije nisu projekcije na stožac ili valjak čije su plohe presječene i razvijene u ravninu, već preslikavanja sfere izravno u ravninu. Iznimke su projekcije koje su definirane kao preslikavanja na površinu stošca, kao što je slučaj s perspektivnim projekcijama.

Konačno, dokazujemo da nije uvijek moguće iz konusne projekcije dobiti odgovarajuću cilindričnu projekciju, kako bi se moglo zaključiti na prvi pogled. Iako vrlo jednostavan, to je najvažniji doprinos ovog članka.

Jednadžbe uspravnih konusnih projekcija obično su zadane u polarnom koordinatnom sustavu u ravnini projekcije:

$$\theta = m\lambda, \ \ \rho = \rho(\varphi), \tag{1}$$

gdje su $\varphi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ i $\lambda \in (-\pi, \pi)$ geografska širina, odnosno dužina, m parametar, $0 < m \le 1$, a θ i ρ polarne koordinate u ravnini projekcije (slika 1).



Slika 2. Točka *T* i njezine pravokutne koordinate *x* i *y*.
Fig. 2 Point *T* and its rectangular coordinates *x* and *y*.

Postavlja se pitanje: kako možemo iz uspravne konusne projekcije, zadane jednadžbama u polarnom sustavu u ravnini projekcije (1), izvesti jednadžbe uspravne cilindrične projekcije u pravokutnom koordinatnom sustavu u ravnini projekcije

$$x = n\lambda, \ \ y = y(\varphi) \tag{2}$$

kao poseban granični slučaj, tako da ostanu sačuvana svojstva konusne projekcije zadane jednadžbama (1). Tražimo cilindričnu projekciju (2) koja je pandan konusne (1). Na primjer, ako je ekvivalentna konusna projekcija zadana jednadžbama (1), kako glase jednadžbe (2) ekvivalentne cilindrične projekcije? U sljedećem poglavlju dat ćemo odgovor na to pitanje za projekcije koje su ekvidistantne, ekvivalentne, konformne i perspektivne projekcije.

Prije toga prisjetimo se da su faktori lokalnih linearnih mjerila uzduž meridijana, odnosno paralela za konusne projekcije sfere polumjera R (Snyder 1987):

$$h(\varphi) = -\frac{d\rho}{R d\varphi}, \ k(\varphi) = \frac{m\rho}{R\cos\varphi} \tag{3}$$

Ako vrijedi $k(\varphi_0)$=1 za neki φ_0, onda prema (3) vrijedi

$$m\rho_0 = R\cos\varphi_0, \tag{4}$$

gdje smo označili $\rho_0 = \rho(\varphi_0)$.

Jednadžbe bilo koje uspravne cilindrične projekcije su (2), gdje su φ i λ geografska širina, odnosno dužina, n

how do the equations (2) of the equal-area cylindrical projection read? We will give the answer to that question in the next sections for equidistant, equal-area, conformal and perspective projections.

Before that, let us recall that the factors of the local linear scale along the meridian and the parallel, respectively, for conic projections of the sphere of radius R are (Snyder 1987)

$$h(\varphi)=-\frac{d\rho}{Rd\varphi},\ k(\varphi)=\frac{m\rho}{R\cos\varphi} \tag{3}$$

If $k(\varphi_0)=1$ holds for some φ_0, then according to (3) we have

$$m\rho_0=R\cos\varphi_0, \tag{4}$$

where we noted $\rho_0=\rho(\varphi_0)$.

The equations of any normal aspect cylindrical projection are (2), where φ and λ are the latitude and longitude, respectively, n the parameter, usually $0<n\le 1$, and x and y the rectangular coordinates in the plane of projection (Figure 2).

If we write simply

$$x=m\lambda,\ y=\rho(\varphi), \tag{5}$$

we obtained the equations of the cylindrical projection from the equations of the conic projection (1) without any problems and without any recalculation. However, it can be easily shown that in this way some properties of the conic projection (1), e.g. equal-area or conformality, will not be preserved. Therefore, the procedure described by relations (5) does not provide the desired solution.

If we substitute $m=1$ in (1), we will get the azimuthal projection equations:

$$\theta=\lambda,\ \rho=\rho(\varphi). \tag{6}$$

If we substitute $m=0$ in (1), we will get

$$\theta=0,\ \rho=\rho(\varphi), \tag{7}$$

which would mean that the entire sphere was mapped to a straight line or part of a straight line. So, to obtain a cylindrical projection from (1) with $m=0$, we must add another condition to prevent the image of the sphere being compressed into a straight line. We can achieve this, for example, by requiring that one parallel that is equidistantly mapped by a conic projection also be equidistantly mapped by a cylindrical projection. In the following sections, we will show how a cylindrical projection can be obtained as a limiting case of a conic one using examples of equidistant along the meridian, equal-area, conformal and central perspective projections.

2 Projections equidistant along meridians

For the normal aspect conic projection of a sphere of radius R given by (1) to be equidistant along the meridians, the condition that the local linear scale factor along the meridians is equal to 1 must be met:

$$h=-\frac{d\rho}{Rd\varphi}=1\,. \tag{8}$$

Integrating equation (8) gives

$$\rho=-R\varphi+C, \tag{9}$$

where C is a constant, $C\ge R\dfrac{\pi}{2}$ to make $\rho\ge 0$ for each value of latitude. So, in the polar coordinate system, the equations of the conic projection that is equidistant along the meridians read:

$$\theta=m\lambda,\ \rho=-R\varphi+C. \tag{10}$$

Let us suppose that φ_0 is the latitude of the equidistantly mapped parallel in that projection. Considering (4), we can write

$$\rho_0=\rho(\varphi_0)=-R\varphi_0+C=\frac{R\cos\varphi_0}{m}, \tag{11}$$

and from there we have

$$C=\frac{R\cos\varphi_0}{m}+R\varphi_0, \tag{12}$$

and then

$$\rho=R\left(\frac{\cos\varphi_0}{m}+\varphi_0-\varphi\right). \tag{13}$$

Following Lambert (1772) and Hinks (1912), let us consider the difference $\rho_0-\rho$. Although both ρ and ρ_0 tend to infinity when $m\to 0$, their difference is finite regardless of m:

$$\rho_0-\rho=R(\varphi-\varphi_0). \tag{14}$$

This allows us to write the equations of a cylindrical projection equidistant along the meridians parametar, obično $0 < n \leq 1$, a x i y pravokutne koordinate u ravnini projekcije (slika 2).

Ako jednostavno napišemo

$$x = m\lambda,\ y = \rho(\varphi), \tag{5}$$

dobit ćemo jednadžbe cilindrične projekcije iz jednadžbi konusne projekcije (1) bez ikakvih problema i preračunavanja. Međutim, lako se može pokazati da na taj način neka svojstva konusne projekcije (1), npr. ekvivalentnost ili konformnost, neće biti sačuvana. Dakle, postupak opisan relacijom (5) ne daje željeno rješenje.

Ako uvrstimo $m = 1$ u (1), dobit ćemo jednadžbe azimutne projekcije:

$$\theta = \lambda,\ \rho = \rho(\varphi). \tag{6}$$

Ako uvrstimo $m = 0$ in (1), dobit ćemo

$$\theta = 0,\ \rho = \rho(\varphi), \tag{7}$$

što bi značilo da je cijela sfera preslikana na pravac ili dio pravca. Dakle, da bismo iz (1) dobili cilindričnu projekciju uz $m = 0$, moramo dodati još jedan uvjet koji će spriječiti da se slika sfere stisne u pravac. To se može postići npr. zahtjevom da jedna paralela koja je ekvidistantno preslikana pri konusnoj projekciji bude također ekvidistantno preslikana pri cilindričnoj projekciji. U sljedećim ćemo poglavljima pokazati kako se može dobiti cilindrična projekcija kao granični slučaj konusne na primjerima ekvidistantne, ekvivalentne, konformne i perspektivne projekcije.

2. Projekcije ekvidistantne uzduž meridijana

Uspravna konusna projekcija sfere polumjera R zadana jednadžbama (1) ekvidistantna je uzduž meridijana ako je lokalno linearno mjerilo uzduž meridijana jednako 1, tj. ako vrijedi

$$h = -\frac{d\rho}{Rd\varphi} = 1 \tag{8}$$

Integriranje jednadžbe (8) daje

$$\rho = -R\varphi + C, \tag{9}$$

gdje je C konstanta, $C \geq R\frac{\pi}{2}$ da bi bilo $\rho \geq 0$ za sve vrijednosti geografske širine. Dakle, u polarnom koordinatnom sustavu jednadžbe konusne projekcije ekvidistantne uzduž meridijana glase:

$$\theta = m\lambda,\ \rho = -R\varphi + C. \tag{10}$$

Pretpostavimo da je φ_0 geografska širina ekvidistantno preslikane paralele u toj projekciji. Uzevši u obzir (4), možemo napisati

$$\rho_0 = \rho(\varphi_0) = -R\varphi_0 + C = \frac{R\cos\varphi_0}{m}, \tag{11}$$

i odatle imamo

$$C = \frac{R\cos\varphi_0}{m} + R\varphi_0, \tag{12}$$

i zatim

$$\rho = R\left(\frac{\cos\varphi_0}{m} + \varphi_0 - \varphi\right). \tag{13}$$

Prema Lambertu (1772) i Hinksu (1912), razmotrimo razliku $\rho_0 - \rho$. Iako i ρ i ρ_0 teže u beskonačnost kad $m \to 0$, njihova je razlika konačna, neovisno o m:

$$\rho_0 - \rho = R(\varphi - \varphi_0). \tag{14}$$

To nam omogućava napisati jednadžbe cilindrične projekcije ekvidistantne uzduž meridijana u pravokutnom koordinatnom sustavu u ravnini projekcije

$$x = n\lambda,\ y = R(\varphi - \varphi_0). \tag{15}$$

Faktori lokalnog linearnog mjerila uzduž meridijana odnosno paralele za cilindrične projekcije sfere radijusa R su (Snyder 1987)

$$h(\varphi) = \frac{dy}{Rd\varphi},\ k(\varphi) = \frac{dx}{R\cos\varphi d\lambda}.$$

Izračunajmo

$$h(\varphi) = \frac{dy}{Rd\varphi} = 1. \tag{16}$$

Dakle, to je cilindrična projekcija ekvidistantna uzduž meridijana. Faktor lokalnog linearnog mjerila za tu projekciju uzduž paralele kojoj odgovara φ_0 je

$$k(\varphi_0) = \frac{n}{R\cos\varphi_0}. \tag{17}$$

Da bi ta paralela bila ekvidistantno preslikana, trebalo bi biti $k(\varphi_0) = 1$, tj.

$$n = R\cos\varphi_0. \tag{18}$$

in a rectangular coordinate system in the projection plane

$$x = n\lambda,\ y = R(\varphi - \varphi_0). \tag{15}$$

The factors of the local linear scale along the meridian and the parallel, respectively, for cylindrical projections of the sphere of radius R are (Snyder 1987)

$$h(\varphi) = \frac{dy}{Rd\varphi},\ k(\varphi) = \frac{dx}{R\cos\varphi d\lambda}.$$

Let us calculate

$$h(\varphi) = \frac{dy}{Rd\varphi} = 1. \tag{16}$$

So, it is a cylindrical projection equidistant along the meridians. The local linear scale factor for that projection along the parallel to which the latitude corresponds φ_0 is

$$k(\varphi_0) = \frac{n}{R\cos\varphi_0}. \tag{17}$$

For this parallel to be equidistantly mapped, $k(\varphi_0) = 1$ should be true, i.e.

$$n = R\cos\varphi_0. \tag{18}$$

Thus, the equations of the normal aspect cylindrical projection equidistant along the meridians, which has the same equidistantly mapped parallel φ_0 as the conic projection (13) which is equidistant along the meridian, are

$$x = R\cos\varphi_0 \cdot \lambda,\ y = R(\varphi - \varphi_0). \tag{19}$$

If we translate the image of the projection by the amount $R\varphi_0$ in the direction of the y axis, we will achieve that the image of the equator is on the coordinate axis x, as is usual in cartographic literature. So, the final equations of the normal aspect cylindrical projection equidistant along the meridian, which has the same equidistantly mapped parallel (φ_0) as the conic projection (13) are

$$x = R\cos\varphi_0 \cdot \lambda,\ y = R\varphi. \tag{20}$$

3 Equal-area projections

For the normal aspect conic projection of a sphere given by (1) to be equal-area, the condition that the product of the factors of the local linear scales along the meridian and along the parallel is equal to 1 must be satisfied, i.e. that

$$hk = -\frac{d\rho}{Rd\varphi}\frac{m\rho}{R\cos\varphi} = 1. \tag{21}$$

Integrating equation (21) gives

$$\rho = R\sqrt{C - \frac{2}{m}\sin\varphi}, \tag{22}$$

where C is a constant. Let us assume that $C \geq \frac{2}{m}$ to make ρ real for each value of latitude. So, in the polar coordinate system, the equations of the equal-area conic projections read:

$$\theta = m\lambda,\ \rho = R\sqrt{C - \frac{2}{m}\sin\varphi}. \tag{23}$$

Let us suppose that φ_0 is the latitude of the equidistantly mapped parallel in that projection. Considering (4), we can write

$$\rho_0 = \rho(\varphi_0) = R\sqrt{C - \frac{2}{m}\sin\varphi_0} = \frac{R}{m}\cos\varphi_0. \tag{24}$$

From (24) it follows

$$C - \frac{2}{m}\sin\varphi_0 = \frac{\cos^2\varphi_0}{m^2}. \tag{25}$$

Now we can calculate

$$\rho_0 - \rho = \frac{2R}{m}\frac{\sin\varphi - \sin\varphi_0}{\sqrt{C - \frac{2}{m}\sin\varphi_0} + \sqrt{C - \frac{2}{m}\sin\varphi}} = \tag{26}$$

$$= 2R\frac{\sin\varphi - \sin\varphi_0}{\cos\varphi_0 + \sqrt{\cos^2\varphi_0 + 2m(\sin\varphi_0 - \sin\varphi)}}.$$

From (26) we obtain

$$\lim_{m\to 0}(\rho_0 - \rho) = R\frac{\sin\varphi - \sin\varphi_0}{\cos\varphi_0}. \tag{27}$$

Now we can write the equations of equal-area cylindrical projection in a rectangular coordinate system in the projection plane

Dakle, jednadžbe uspravne cilindrične projekcije ekvidistantne uzduž meridijana koja ima istu ekvidistantno preslikanu paralelu (φ_0) kao konusna projekcija (13) su

$$x = R\cos\varphi_0 \cdot \lambda\,,\; y = R(\varphi - \varphi_0). \tag{19}$$

Pomaknemo li sliku projekcije za iznos $R\varphi_0$ u smjeru osi y, postići ćemo da slika ekvatora bude na koordinatnoj osi x kao što je uobičajeno u kartografskoj literaturi. Dakle, konačne jednadžbe uspravne cilindrične projekcije ekvidistantne uzduž meridijana koja ima istu ekvidistantno preslikanu paralelu (φ_0) kao konusna projekcija (13) su

$$x = R\cos\varphi_0 \cdot \lambda\,,\; y = R\varphi. \tag{20}$$

3. Ekvivalentne projekcije

Uspravna konusna projekcija sfere zadana jednadžbama (1) ekvivalentna je ako je produkt faktora lokalnog mjerila uzduž meridijana i uzduž paralele jednak 1, tj. ako vrijedi

$$hk = -\frac{d\rho}{Rd\varphi}\frac{m\rho}{R\cos\varphi} = 1. \tag{21}$$

Integriranjem jednadžbe (21) daje

$$\rho = R\sqrt{C - \frac{2}{m}\sin\varphi}, \tag{22}$$

gdje je C konstanta. Pretpostavimo da je $C \geq \frac{2}{m}$ kako bi ρ bio realan broj za svaku vrijednost geografske širine. Dakle, u polarnom koordinatnom sustavu jednadžbe ekvivalentne konusne projekcije glase:

$$\theta = m\lambda,\; \rho = R\sqrt{C - \frac{2}{m}\sin\varphi}. \tag{23}$$

Pretpostavimo da je φ_0 geografska širina ekvidistantno preslikane paralele pri toj projekciji. Uzevši u obzir (4), možemo napisati

$$\rho_0 = \rho(\varphi_0) = R\sqrt{C - \frac{2}{m}\sin\varphi_0} = \frac{R}{m}\cos\varphi_0. \tag{24}$$

Iz (24) slijedi

$$C - \frac{2}{m}\sin\varphi_0 = \frac{\cos^2\varphi_0}{m^2}. \tag{25}$$

Sad možemo izračunati

$$\rho_0 - \rho = \frac{2R}{m}\frac{\sin\varphi - \sin\varphi_0}{\sqrt{C - \frac{2}{m}\sin\varphi_0} + \sqrt{C - \frac{2}{m}\sin\varphi}} =$$

$$= 2R\frac{\sin\varphi - \sin\varphi_0}{\cos\varphi_0 + \sqrt{\cos^2\varphi_0 + 2m(\sin\varphi_0 - \sin\varphi)}}. \tag{26}$$

Iz (26) dobijemo

$$\lim_{m\to 0}(\rho_0 - \rho) = R\frac{\sin\varphi - \sin\varphi_0}{\cos\varphi_0}. \tag{27}$$

Sad možemo napisati jednadžbe ekvivalentne cilindrične projekcije u pravokutnom koordinatnom sustavu u ravnini projekcije

$$x = n\lambda,\; y = R\frac{\sin\varphi - \sin\varphi_0}{\cos\varphi_0}. \tag{28}$$

Ako želimo da paralela kojoj odgovara geografska širina φ_0 bude ekvidistantno preslikana, tada mora biti $n = R\cos\varphi_0$ kao što je pokazano u prethodnom poglavlju (formula (18)). Dakle, konačne jednadžbe ekvidistantne cilindrične projekcije su:

$$x = R\cos\varphi_0 \cdot \lambda,\; y = R\frac{\sin\varphi - \sin\varphi_0}{\cos\varphi_0}. \tag{29}$$

Provjerimo

$$h(\varphi) = \frac{\cos\varphi}{\cos\varphi_0},\; k(\varphi) = \frac{\cos\varphi_0}{\cos\varphi} \tag{30}$$

Dakle,

$$h(\varphi)k(\varphi) = 1. \tag{31}$$

Pomaknemo li sliku projekcije za iznos $R\tan\varphi_0$ u smjeru osi y, postići ćemo da slika ekvatora bude na koordinatnoj osi x kao što je uobičajeno u kartografskoj literaturi. Dakle, konačne jednadžbe uspravne ekvivalentne cilindrične projekcije koja ima istu ekvidistantno preslikanu paralelu (φ_0) kao konusna projekcija (23) su

$$x = R\cos\varphi_0 \cdot \lambda,\; y = R\frac{\sin\varphi}{\cos\varphi_0}. \tag{32}$$

4. Konformne projekcije

Da bi uspravna konusna projekcija zadana s (1) bila konformna, mora biti ispunjen uvjet da su faktori lokalnih mjerila uzduž meridijana i paralela međusobno jednaki, tj. da vrijedi:

$$-\frac{d\rho}{Rd\varphi} = \frac{m\rho}{R\cos\varphi}. \tag{33}$$

Integriranje jednadžbe (33) daje

$$\rho = C\tan^m\left(\frac{\pi}{4} - \frac{\varphi}{2}\right), \tag{34}$$

$$x = n\lambda, \quad y = R\frac{\sin\varphi - \sin\varphi_0}{\cos\varphi_0}. \tag{28}$$

If we want the parallel corresponding to the latitude φ_0 to be equidistantly mapped, then it should be $n = R\cos\varphi_0$, as we showed in the previous section (formula (18)). So, the final equations of the equal-area cylindrical projection are:

$$x = R\cos\varphi_0 \cdot \lambda, \; y = R\frac{\sin\varphi - \sin\varphi_0}{\cos\varphi_0}. \tag{29}$$

Let us check

$$h(\varphi) = \frac{\cos\varphi}{\cos\varphi_0}, \; k(\varphi) = \frac{\cos\varphi_0}{\cos\varphi} \tag{30}$$

Therefore,

$$h(\varphi)k(\varphi) = 1. \tag{31}$$

If we translate the image of the projection by the amount $R\tan\varphi_0$ in the direction of the y axis, we will achieve that the image of the equator is on the coordinate axis x, as is usual in cartographic literature. So, the final equations of the normal aspect equal-area cylindrical projection, which has the same equidistantly mapped parallel φ_0 as the conic projection (23) are

$$x = R\cos\varphi_0 \cdot \lambda, \; y = R\frac{\sin\varphi}{\cos\varphi_0}. \tag{32}$$

4 Conformal projections

For the normal aspect conic projection given by (1) to be conformal, the condition is that the factors of the local linear scales along the meridians and along the parallels are equal, i.e. that the following holds

$$-\frac{d\rho}{Rd\varphi} = \frac{m\rho}{R\cos\varphi}. \tag{33}$$

Integrating equation (33) gives

$$\rho = C\tan^m\left(\frac{\pi}{4} - \frac{\varphi}{2}\right), \tag{34}$$

where $C > 0$ is a constant.

So, in the polar coordinate system, the equations of the conformal conic projections read:

$$\theta = m\lambda, \; \rho = C\tan^m\left(\frac{\pi}{4} - \frac{\varphi}{2}\right). \tag{35}$$

Let us suppose that φ_0 is the latitude of the equidistantly mapped parallel in that projection. Considering (4), we can write

$$\rho_0 = \rho(\varphi_0) = C\tan^m\left(\frac{\pi}{4} - \frac{\varphi_0}{2}\right) = \frac{R}{m}\cos\varphi_0. \tag{36}$$

From (36) it follows

$$C = \frac{R\cos\varphi_0}{m\tan^m\left(\frac{\pi}{4} - \frac{\varphi_0}{2}\right)}. \tag{37}$$

Now we can calculate

$$\rho_0 - \rho = \frac{R\cos\varphi_0}{m\tan^m\left(\frac{\pi}{4} - \frac{\varphi_0}{2}\right)}\left[\tan^m\left(\frac{\pi}{4} - \frac{\varphi_0}{2}\right) - \tan^m\left(\frac{\pi}{4} - \frac{\varphi}{2}\right)\right] =$$

$$= \frac{R\cos\varphi_0}{m}\left[1 - \frac{\tan^m\left(\frac{\pi}{4} - \frac{\varphi}{2}\right)}{\tan^m\left(\frac{\pi}{4} - \frac{\varphi_0}{2}\right)}\right] = \tag{38}$$

$$= R\cos\varphi_0 \frac{1 - \left[\tan\left(\frac{\pi}{4} + \frac{\varphi_0}{2}\right)\tan\left(\frac{\pi}{4} - \frac{\varphi}{2}\right)\right]^m}{m\tan^m\left(\frac{\pi}{4} - \frac{\varphi_0}{2}\right)}.$$

When $m \to 0$ then the last fraction is of the form 0/0. Applying l'Hôpital's rule, we will get

$$\lim_{m\to 0}(\rho_0 - \rho) = -R\cos\varphi_0 \ln\left[\tan\left(\frac{\pi}{4} + \frac{\varphi_0}{2}\right)\tan\left(\frac{\pi}{4} - \frac{\varphi}{2}\right)\right]. \tag{39}$$

Now we can write the equations of the conformal cylindrical projection in the rectangular coordinate system in the projection plane

$$x = n\lambda,$$
$$y = R\cos\varphi_0 \ln\left[\tan\left(\frac{\pi}{4} + \frac{\varphi_0}{2}\right)\tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)\right]. \tag{40}$$

If we want the parallel corresponding to the latitude φ_0 to be equidistantly mapped, then it should be $n = R\cos\varphi_0$ (formula (18)). The equations of the conformal cylindrical projection are:

gdje je C > 0 konstanta.

Dakle, u polarnom koordinatnom sustavu jednadžbe konformne konusne projekcije su:

$$\theta = m\lambda,\ \rho = C\tan^m\left(\frac{\pi}{4}-\frac{\varphi}{2}\right). \tag{35}$$

Pretpostavimo da je φ_0 geografska širina ekvidistantno preslikane paralele u toj projekciji. Uzevši u obzir (4), možemo napisati

$$\rho_0 = \rho(\varphi_0) = C\tan^m\left(\frac{\pi}{4}-\frac{\varphi_0}{2}\right) = \frac{R}{m}\cos\varphi_0. \tag{36}$$

Iz (36) slijedi

$$C = \frac{R\cos\varphi_0}{m\tan^m\left(\frac{\pi}{4}-\frac{\varphi_0}{2}\right)}. \tag{37}$$

Sad možemo izračunati

$$\rho_0-\rho = \frac{R\cos\varphi_0}{m\tan^m\left(\frac{\pi}{4}-\frac{\varphi_0}{2}\right)}\left[\tan^m\left(\frac{\pi}{4}-\frac{\varphi_0}{2}\right)-\tan^m\left(\frac{\pi}{4}-\frac{\varphi}{2}\right)\right]=$$

$$= \frac{R\cos\varphi_0}{m}\left[1-\frac{\tan^m\left(\frac{\pi}{4}-\frac{\varphi}{2}\right)}{\tan^m\left(\frac{\pi}{4}-\frac{\varphi_0}{2}\right)}\right]= \tag{38}$$

$$= R\cos\varphi_0\frac{1-\left[\tan\left(\frac{\pi}{4}+\frac{\varphi_0}{2}\right)\tan\left(\frac{\pi}{4}-\frac{\varphi}{2}\right)\right]^m}{m\tan^m\left(\frac{\pi}{4}-\frac{\varphi_0}{2}\right)}.$$

Kad $m\to 0$, tada je posljednji razlomak oblika 0/0. Primjenom l'Hôpitalova pravila dobit ćemo

$$\lim_{m\to 0}(\rho_0-\rho) = -R\cos\varphi_0\ln\left[\tan\left(\frac{\pi}{4}+\frac{\varphi_0}{2}\right)\tan\left(\frac{\pi}{4}-\frac{\varphi}{2}\right)\right]. \tag{39}$$

Sad možemo napisati jednadžbe konformne cilindrične projekcije u pravokutnom koordinatnom sustavu u ravnini projekcije

$$x = n\lambda,$$
$$y = R\cos\varphi_0\ln\left[\tan\left(\frac{\pi}{4}+\frac{\varphi_0}{2}\right)\tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)\right]. \tag{40}$$

Ako želimo da paralela kojoj odgovara geografska širina φ_0 bude ekvidistantno preslikana, tada mora biti $n = R\cos\varphi_0$ kao što je prije pokazano. Jednadžbe konformne cilindrične projekcije su:

$$x = R\cos\varphi_0\cdot\lambda,$$
$$y = R\cos\varphi_0\ln\left[\tan\left(\frac{\pi}{4}+\frac{\varphi_0}{2}\right)\tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)\right]. \tag{41}$$

Provjerimo za projekciju (41):

$$h(\varphi) = \frac{dy}{d\varphi} = \frac{\cos\varphi_0}{\cos\varphi},\ k(\varphi) = \frac{\cos\varphi_0}{\cos\varphi}. \tag{42}$$

Dakle,

$$h(\varphi) = k(\varphi). \tag{43}$$

Pomaknemo li sliku projekcije za iznos

$$R\cos\varphi_0\ln\tan\left(\frac{\pi}{4}+\frac{\varphi_0}{2}\right)$$

u smjeru osi y, postići ćemo da slika ekvatora bude na koordinatnoj osi x kao što je uobičajeno u kartografskoj literaturi. Dakle, konačne jednadžbe uspravne konformne cilindrične projekcije koja ima istu ekvidistantno preslikanu paralelu (φ_0) kao konusna projekcija (34) su

$$x = R\cos\varphi_0\cdot\lambda,\ y = R\cos\varphi_0\ln\tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right). \tag{44}$$

5. Perspektivne projekcije

Jednadžbe uspravne perspektivne projekcije na konus mogu se napisati u polarnom koordinatnom sustavu u obliku (Lapaine i Frančula 1992):

$$\theta = \sin\alpha\cdot\lambda,\ \rho = d\,[\cot\alpha-\tan(\varphi-\alpha)], \tag{45}$$

gdje je α parametar koji se geometrijski može opisati kao polovina kuta u vrhu konusa na koji se sfera preslikava, a d udaljenost plašta stošca od središta (slika 3).

Napišimo

$$\rho_0 = \rho(\varphi_0) = d\,[\cot\alpha-\tan(\varphi_0-\alpha)] \tag{46}$$

i izračunajmo

$$\rho_0-\rho = d[\tan(\varphi-\alpha)-\tan(\varphi_0-\alpha)]. \tag{47}$$

Iako i ρ i ρ_0 teže u beskonačnost kad $\alpha\to 0$, njihova je razlika konačna

$$\lim_{\alpha\to 0}(\rho_0-\rho) = d(\tan\varphi-\tan\varphi_0). \tag{48}$$

$$x = R\cos\varphi_0 \cdot \lambda,$$
$$y = R\cos\varphi_0 \ln\left[\tan\left(\frac{\pi}{4}+\frac{\varphi_0}{2}\right)\tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)\right]. \quad (41)$$

Let us check for the projection (41):

$$h(\varphi) = \frac{dy}{d\varphi} = \frac{\cos\varphi_0}{\cos\varphi},\ k(\varphi) = \frac{\cos\varphi_0}{\cos\varphi}. \quad (42)$$

Therefore,

$$h(\varphi) = k(\varphi). \quad (43)$$

If we translate the image of the projection by the amount

$$R\cos\varphi_0 \ln\tan\left(\frac{\pi}{4}+\frac{\varphi_0}{2}\right)$$

in the direction of the y axis, we will achieve that the image of the equator is on the coordinate axis x, as is usual in cartographic literature. So, the final equations of the normal aspect conformal cylindrical projection, which has the same equidistantly mapped parallel φ_0 as the conic projection (34) are

$$x = R\cos\varphi_0 \cdot \lambda,\ y = R\cos\varphi_0 \ln\tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right). \quad (44)$$

5 Perspective projections

The equations of the normal aspect central perspective projection on the cone can be written in the polar coordinate system in the form (Lapaine, Frančula 1992):

$$\theta = \sin\alpha \cdot \lambda,\ \ \rho = d\,[\cot\alpha - \tan(\varphi-\alpha)\,], \quad (45)$$

where α is a parameter that can be described geometrically as half the angle at the apex of the cone onto which the sphere is mapped and d the distance of the surface of the cone from the center (Figure 3).

Let us write

$$\rho_0 = \rho(\varphi_0) = d\,[\cot\alpha - \tan(\varphi_0-\alpha)] \quad (46)$$

and calculate

$$\rho_0 - \rho = d[\tan(\varphi-\alpha) - \tan(\varphi_0-\alpha)]. \quad (47)$$

If $\alpha\to 0$, then ρ and ρ_0 tend to infinity, but their difference is finite

$$\lim_{\alpha\to 0}(\rho_0-\rho) = d(\tan\varphi - \tan\varphi_0). \quad (48)$$

For the parallel corresponding to the latitude φ_0 to be equidistantly mapped onto the cone according to formulas (45), considering (4) it should be true

$$\rho_0 = \frac{R}{\sin\alpha}\cos\varphi_0. \quad (49)$$

From (46) and (49) it follows

$$\cos(\varphi_0 - \alpha) = \frac{d}{R}. \quad (50)$$

From (50) we can conclude that an equidistant mapped parallel in perspective conic projection exists if $d \le R$ and that $d = R\cos\varphi_0$ holds for $\alpha = 0$.

If we want the parallel corresponding to the latitude φ_0 to be equidistantly mapped in the cylindrical projection, then it must be $n = R\cos\varphi_0$, as shown before.

Thus, the equations of the normal aspect central perspective on the cylinder in the rectangular coordinate system in the plane are

$$x = R\cos\varphi_0 \cdot \lambda,$$
$$y = R\cos\varphi_0\,(\tan\varphi - \tan\varphi_0). \quad (51)$$

If we translate the image of the projection by the amount $R\sin\varphi_0$ in the direction of the y axis, we will achieve that the image of the equator is on the coordinate axis x, as is usual in cartographic literature. So, the final equations of the normal aspect perspective cylindrical projection, which has the same equidistantly mapped parallel (φ_0) as the perspecitve conic projection (45) reads

$$x = R\cos\varphi_0 \cdot \lambda,$$
$$y = R\cos\varphi_0\tan\varphi. \quad (52)$$

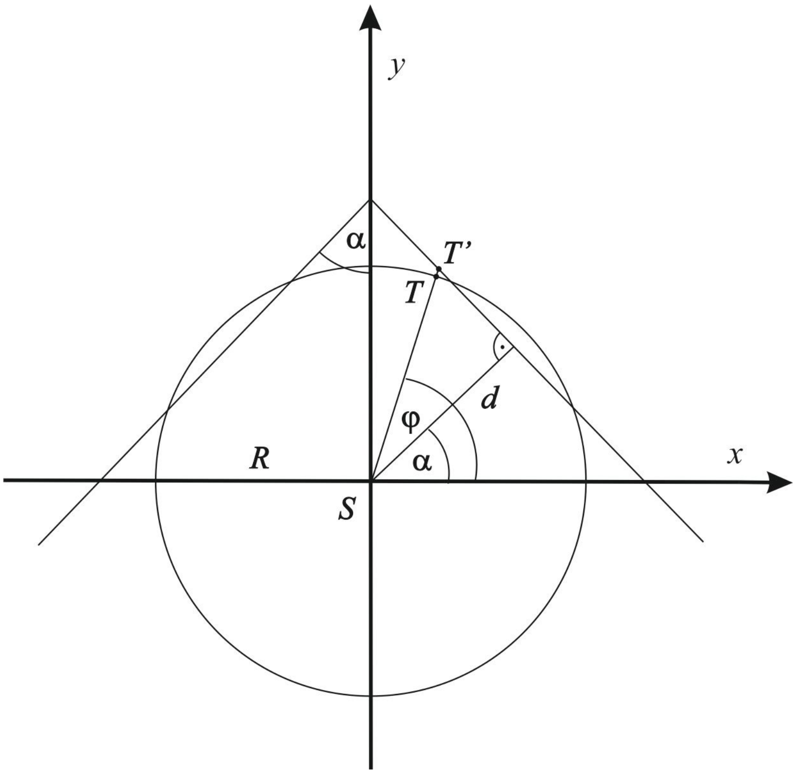
6 Projections equidistant along parallels

For the normal aspect conic projection of the sphere of radius R given by (1) to be equidistant along the parallels, the condition that the local linear scale factor along the parallels is equal to 1 must be met:

$$k = \frac{m\rho}{R\cos\varphi} = 1. \quad (53)$$

From equation (51) the following is immediately obtained

$$\rho = \frac{R}{m}\cos\varphi. \quad (54)$$



Slika 3. Gnomonska perspektivna konusna projekcija. Točka *T'* na plohi konusa je slika točke *T* na sferi polumjera *R*. Točke *S*, *T* i *T'* su kolinearne. Vidi (Lapaine, Frančula 1992).

Fig. 3 The gnomonic perspective conic projection. Point *T'* on the surface of the cone is the image of point *T* on the sphere of radius *R*. Points *S*, *T* and *T'* are collinear. See (Lapaine, Frančula 1992).

Da bi paralela kojoj odgovara geografska širina φ_0 bila ekvidistantno preslikana na konus po formulama (45), treba prema (4) vrijediti

$$\rho_0 = \frac{R}{\sin\alpha}\cos\varphi_0. \tag{49}$$

Iz (46) i (49) slijedi

$$\cos(\varphi_0 - \alpha) = \frac{d}{R}. \tag{50}$$

Iz (50) možemo zaključiti da ekvidistantno preslikana paralela pri perspektivnoj konusnoj projekciji postoji ako je $d \le R$ i da za $\alpha = 0$ vrijedi $d = R\cos\varphi_0$.

Ako želimo da paralela kojoj odgovara geografska širina φ_0 bude ekvidistantno preslikana pri cilindričnoj projekciji, tada mora biti $n = R\cos\varphi_0$, kao što je prije pokazano.

Dakle, jednadžbe uspravne centralne ili gnomonske cilindrične perspektivne projekcije u pravokutnom sustavu u ravnini projekcije glase

$$x = R\cos\varphi_0 \cdot \lambda,$$
$$y = R\cos\varphi_0 (\tan\varphi - \tan\varphi_0). \tag{51}$$

Pomaknemo li sliku projekcije za iznos $R\sin\varphi_0$ u smjeru osi y, postići ćemo da slika ekvatora bude na koordinatnoj osi x kao što je uobičajeno u kartografskoj literaturi. Dakle, konačne jednadžbe uspravne konformne cilindrične projekcije koja ima istu ekvidistantno preslikanu paralelu (φ_0) kao konusna projekcija (45) su

$$x = R\cos\varphi_0 \cdot \lambda,$$
$$y = R\cos\varphi_0 \tan\varphi. \tag{52}$$

6. Projekcije ekvidistantne uzduž paralela

Da bi uspravna konusna projekcija sfere polumjera R zadana s (1) bila ekvidistantna uzduž paralela, faktor lokalnog linearnog mjerila uzduž paralela mora biti jednak 1:

$$k = \frac{m\rho}{R\cos\varphi} = 1. \tag{53}$$

Iz (53) neposredno slijedi

$$\rho = \frac{R}{m}\cos\varphi. \tag{54}$$

Let us write

$$\rho_0 = \rho(\varphi_0) = \frac{R}{m}\cos\varphi_0 \qquad (55)$$

and calculate

$$\rho_0 - \rho = \frac{R}{m}(\cos\varphi_0 - \cos\varphi). \qquad (56)$$

When $m \to 0$ then $\rho_0 - \rho \to \infty$. In the theory of map projections, it is usually assumed that the functions defining map projections are real, single-valued, continuous, and differentiable functions of φ and λ in some domain and that their Jacobian determinant does not vanish (Tobler 1962). Therefore, in the described way, a cylindrical projection equidistant along the parallels cannot be obtained as a limiting case of a conic projection equidistant along parallels. Such a projection does not exist at all because in all normal aspect cylindrical projections all parallels are of equal length, and this is not the case on a sphere. This example proves that not all conic projections have a cylindrical limiting case as it seems at first glance.

7 Conclusion

Lambert (1772) derived the formula for the conformal conic projection. In the same publication, Lambert derived the equation of the Mercator projection as a limiting case of a conformal conic projection. Hinks (1912) used the same method as Lambert to construct a cylindrical projection as a limiting case of a conic projection. From the conic equidistant along the meridians (simple conic) he derives the cylindrical equidistant projection. For a simple equal-area projection with one standard parallel, Hinks does not give a derivation.

In this article, we give a derivation for the equal-area cylindrical projection as a limiting case of the equal-area conic projection. In addition, we give a derivation for the central perspective cylindrical projection as a limiting case of the central conic projection.

Furthermore, Hinks implies in his derivations that a conic projection is a mapping on a cone and that a cylindrical projection is a mapping on a cylinder. In this article, the conic and cylindrical projections are not projections on a cone or a cylinder whose surfaces are cut and developed into a plane, but mappings of the sphere directly into the plane. Exceptions are projections that are defined as mappings on the surface of a cone or plane, as is the case with perspective projections.

The main result of this paper is that it is not always possible to obtain a corresponding cylindrical projection as a limiting case from a conic projection, as one might conclude at first glance, and no one has noticed this so far. So, it is not advisable to interpret cylindrical projections as limiting cases of conic projections.

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Napišimo

$$\rho_0 = \rho(\varphi_0) = \frac{R}{m}\cos\varphi_0 \tag{55}$$

i izračunajmo

$$\rho_0 - \rho = \frac{R}{m}(\cos\varphi_0 - \cos\varphi). \tag{56}$$

Kad $m \to 0$ tada $\rho_0 - \rho \to \infty$. U teoriji kartografskih projekcija obično se pretpostavlja da su funkcije koje definiraju kartografske projekcije realne, neprekidne i diferencijabilne funkcije od φ i λ na nekom području i da njihova Jacobijeva determinanta ne isčezava (Tobler 1962). Dakle, na opisani se način ne može dobiti cilindrična projekcija ekvidistantna uzduž paralela kao granični slučaj konusne projekcije ekvidistantne uzduž paralela. Takva projekcija ne može se dobiti ni na koji način, ona ne postoji jer su kod svih uspravnih cilindričnih projekcija slike paralela jednake duljine, a paralele na sferi nisu. Ovaj primjer pokazuje da nemaju sve konusne projekcije granični cilindrični pandan, kao što se na prvi pogled čini.

7. Zaključak

Lambert (1772) je izveo formulu za konformnu konusnu projekciju. U istoj publikaciji Lambert je izveo jednadžbu Mercatorove projekcije kao graničnog slučaja konformne konusne projekcije. Hinks (1912) se koristio istom metodom kao Lambert da bi konstruirao cilindričnu projekciju kao granični slučaj konusne. Iz konusne ekvidistantne uzduž mjeridijana (jednostavna konusna) izveo je cilindričnu ekvidistantnu projekciju. Za jednostavnu ekvivalentnu projekciju s jednom standardnom paralelom Hinks nije dao izvod.

U ovom članku izvodimo ekvivalentnu cilindričnu projekciju kako granični slučaj ekvivalentne konusne projekcije. Uz to izvodimo centralnu perspektivnu cilindričnu projekciju kao granični slučaj centralne konusne projekcije.

Nadalje, Hinks u svojim izvodima pretpostavlja da je konusna projekcija preslikavanje na konus i da je cilindrična projekcija preslikavanje na cilindar. U ovom članku konusne i cilindrične projekcije nisu projekcije na konus, odnosno cilindar, čije se plohe razrežu i razviju u ravninu, nego preslikavanja sfere izravno u ravninu. Izuzetak su projekcije koje su definirane kao preslikavanja na plohu konusa ili u ravninu, kao što je slučaj kod perspektivnih projekcija.

Glavni je rezultat ovog članka da nije uvijek moguće dobiti odgovarajuću cilindričnu projekciju kao granični slučaj konusne projekcije, kao što bi se moglo zaključiti na prvi pogled, a nitko to do sada nije uočio. Dakle, nije preporučljivo interpretirati cilindrične projekcije kao granične slučajeve konusnih projekcija.

Zahvala

Zahvaljujem anonimnim recenzentima na vrijednim komentarima. Njihove su mi primjedbe omogućile znatno poboljšanje prve inačice rukopisa.

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